USING TEST ANSWERS TO INTERPRET ASPECTS OF THE MATHEMATICAL PROFICIENCY OF PRESERVICE PRIMARY SCHOOL TEACHERS

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This paper analyses preservice primary teachers' test answers to understand the aspects of mathematical proficiency evident in their written work. It reports on the nature of strategic competence and adaptive reasoning acts of a group of first-year education students who took a semester course in problem-solving that foregrounded the processes of mathematical reasoning and discussion. A conceptual framework was developed that relates strategic competence to decisions about formatting moves, and adaptive reasoning to decisions about specialisation and generalisation moves. This paper argues that students who moved flexibly between formal mathematical structuring and emergent mathematical structuring during problem-solving show effective strategic competence and adaptive reasoning. We conclude that such reasoning moves should be made explicit in education courses, as aspects of mathematical proficiency.

Keywords: Pre-service teachers, mathematics proficiency, problem-solving, adaptive reasoning, strategic competence.

INTRODUCTION

Prospective primary mathematics teachers naturally expect to use their high school knowledge when they solve problems. However, the algebraic high school processes and procedures by themselves are not appropriate for primary school classrooms. We reason that if preservice teachers (PTs) revert to numerical and pre-algebraic approaches to verify their algebraic solutions, they access structures and reasoning that can inform their primary school teaching. With this in mind, we analysed student test solutions from a mathematics course designed to influence and develop PTs' mathematical proficiency through problem-solving. The course focused on using and discussing numerical, geometric, and algebraic structuring and reasoning that is appropriate for progression from primary school to high school. In this first-year mathematics education course for prospective primary school teachers, the focus was on providing rich experiences of 'doing mathematics' that encouraged PTs to compare their reasoning and problem-solving processes. The problem-solving module took place over 10 weeks, for 90 minutes once per week at the beginning of the academic year. There were 51 students who are studying to teach mathematics in the Intermediate Phase (learners aged 9−13). The PTs were expected to solve different problems across a variety of content topics from the Intermediate Phase school curriculum:

numbers, patterns, geometry, and measurement. In class, they worked on the problems both as individuals and in groups.

The PTs' high school backgrounds varied greatly but cohere around education practices where procedural fluency carried high importance in assessments, and where discussion of their own reasoning and habits in classrooms were scarce. This course could be seen as a disruption of their previous beliefs and habits about mathematics discourse. The pedagogy of the course asked PTs to reflect on their own school knowledge and habits and challenged them to communicate their strategies and justifications in words, diagrams, and mathematical text.

The test solutions provided us with an opportunity to engage with their mathematical knowledge and reasoning on split levels – appropriate for high school and primary school. We analysed their solutions through the lens of strategic competence and adaptive reasoning, and the emergent and mathematical structure of their written work. We were interested in whether students who provided only algebraic solutions (some incorrect) have recourse to numerical and geometric reasoning to verify their solutions; and whether students who only provided numerical or geometric solutions could generalise their processes and procedures with the use of algebra.

As teacher educators, we analysed and reflected on students' written solutions to answer the question: What do first-year primary school PTs' test solutions reveal about the nature of their strategic competence and adaptive reasoning?

CONCEPTUAL FRAMEWORK

This study is guided by two different frameworks to better understand the nature of the students' solutions. The first is the mathematical proficiency framework (Kilpatrick et al., 2001) which comprises five strands: procedural fluency, conceptual understanding, strategic competence, adaptive reasoning, and productive disposition. We focus specifically on strategic competence and adaptive reasoning as strands that draw on mathematical structure and provide opportunities for teachers to elicit and model 'doing mathematics' at different stages of development.

Strategic competence is summarised by Kilpatrick et al. (2001) as follows: "– the ability to formulate, represent, and solve mathematical problems" (p. 116). In relation to the other strands and particularly to adaptive reasoning, strategic competence is about the realisation of "any mental representation that maintains the structural relations among the variables of the problem" (p. 116), and such realisations can be viewed on a continuum of sophistication (p. 126). Structural relations are usually visually represented in calculation plans, and these are what teachers must utilise to engage with learner reasoning. A teacher who wants to promote strategic competence must be capable of "not only com[ing] up with several approaches to a non-routine problem ... but could also choose flexibly among reasoning, guess-and-check, algebraic, or other methods ..." (p. 127). Adaptive reasoning is "– capacity for logical thought, reflection, explanation, and justification" (Kilpatrick et al., 2002, p. 116). As a personal reasoning skill, it is "… a meta-cognitive skill that one uses to figure out if one is on the right track and whether there is consistency in what one is doing and to evaluate one's answers" (p. 133). If teachers are not aware of alternative ways to structure a situation while reflecting on their reasoning, they are unlikely to talk to learners in ways that promote adaptive reasoning.

Our second framework draws from the work of Venkat et al. (2019) which describes the architecture of mathematical structure as 'formatting moves' and 'generalising moves' (Figure 1), which can be seen as different levels of sophistication of structuring. Venkat et al. (2019) distinguish between emergent structure as a result of formatting actions that involve analysing local relationships, and mathematical structure that results from formatting actions involving generic or general relationships. Note that in this framework, the formatting actions proceed in one direction only, from the case to the structure, while generalising and specialising actions create bridges between cases and structures in both directions. We use their framework as a mirror to the split-level reasoning that primary school teachers must employ – knowing formal structures (such as algebra) but teaching through emergent structures (such as numeric).

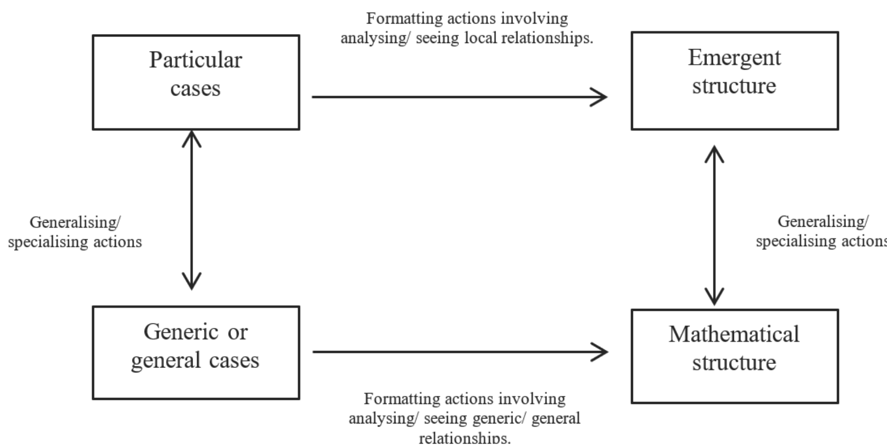


Figure 1: Structures and formatting actions in Mathematics (Venkat et al., 2019, p. 16)

LITERATURE REVIEW

Star and Rittle-Johnson (2008) point out that students who struggle with algebra often use insufficient strategies because they do not notice the relevant structures. This adds weight to the argument that awareness of, and time to reflect on, mathematical structures such as explicated by Venkat et al. (2019) is an important foundation for algebra proficiency (Chan et al., 2022). Discussions of Mathematical Proficiency in teacher education tend to focus primarily on improving aspects of instruction to include all strands (Groth, 2017). According to a review of literature by Corrêa and Haslam (2020) there are few studies related to mathematical proficiency as the basis to design assessments. However, contrary to arguments that mathematical proficiency cannot be assessed through multiple-choice tests, Khairani and Nordin (2011) created such a test to assess the conceptual understanding, procedural fluency, and strategic competence of 588 high school students and concluded that proficiency factored equally into the three strands they targeted. An assessment item from a written test was used to analyse and reflect on the relationship between PTs' strategic competence and adaptive reasoning through the ways in which they format and structure mathematics in their solutions. Procedural fluency was assumed to be in place, since the tasks were focused on non-routine problems that could be posed to primary school learners.

DESIGN AND METHODOLOGY

At the end of the semester course, the PTs were given a summative test with six problem questions, including familiar and unfamiliar contexts. They were instructed to provide written explanations of their reasoning. It was noticed that in this assessment situation, the PTs seemed to use the two different ways of structuring described by Venkat et al. (2019) for different purposes. We related strategic competence with the formatting moves the PTs made since they describe conscious decisions about ways to approach a problem. However, their written work suggests that as strategic actions, the formatted structures were sometimes unpacked as (reversed to) cases. Adaptive reasoning was evident from generalising moves, as we noticed that PTs who checked their solutions, or who got stuck while solving a problem, tended to use generalising or specialising actions to access the opposite formatting action than they started with.

Each problem was analysed according to the following guide: What strategy did the PT use to start the problem-solving process (coded as **S**trategy **F**ormat **T**op axis, or **S**trategy **F**ormat **B**ottom axis)? How did the PT change the strategy, if it was changed (Coded as **A**dapt **M**athematical to **E**mergent or **A**dapt **E**mergent to **M**athematical)? If there was no change in strategy, we coded "No adaptation". How did the PT proceed (Coded as **A**dapt **G**eneric Cases to **P**articular Cases or **A**dapt **P**articular Cases to **G**eneric Cases)?

The following adaptations to the Venkat et al. (2019) framework emerged:

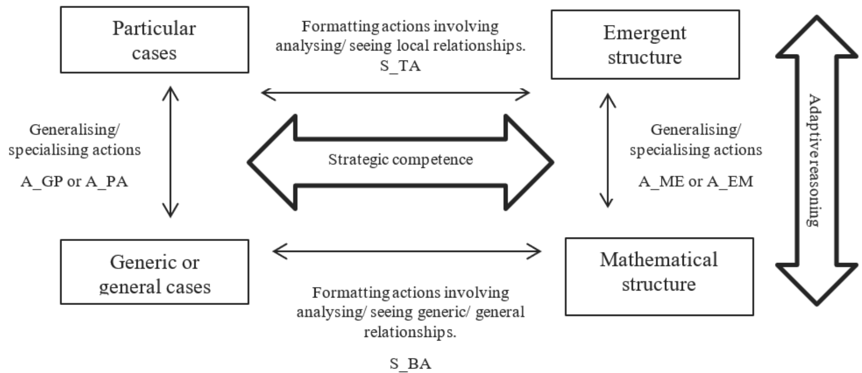


Figure 2: Structures and formatting actions as indications of Strategic competence and Adaptive reasoning of PTs (adapted from Venkat et al., 2019, p. 16)

With reference to Figure 2, it is proposed that the way the PTs structure mathematical relationships through the two kinds of formatting moves are instances of strategic competence when solving problems. Adaptive reasoning is then evident in the use of alternative structures as they control and check their solution processes. For example, an algebraic solution can be checked by specialising a numerical structure for comparison with the use of particular cases. In Figure 2, the top and bottom arrows are bi-directional in response to our analysis that PTs sometimes refer back from a structure under construction to the cases used to construct it.

The next section presents one of the test items and three different PTs' responses to understand how the structure framework helps to make sense of students' strategic competence and adaptive reasoning moves.

RESULTS

This is a report on our analysis of one problem, which calls on PTs to refer to a familiar mathematical structure and asks them to use formatting actions involving generic relationships. A screenshot of the PT's work is provided with an indication of the trajectory of their actions with an arrow on the adapted structure diagram of Venkat et al. (2019, p. 16). The bold arrows indicate the axis on which the PT's moves can be placed.

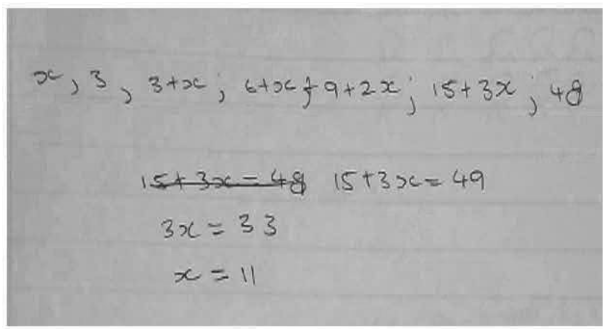
Test item sample

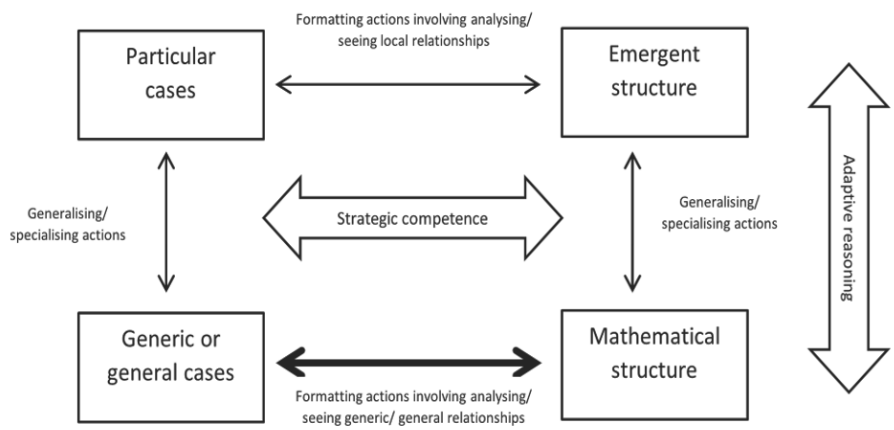
Study the sequence of numbers 1; 2; 3; 5; 8; 13…….

Note that 1 + 2 = 3, 2 + 3 = 5, 3 + 5 = 8 and so on. Use the same rule to complete the sequence below using algebra as a problem-solving strategy.

__, 3, __, __, __, __, 49

Solution: PT 1

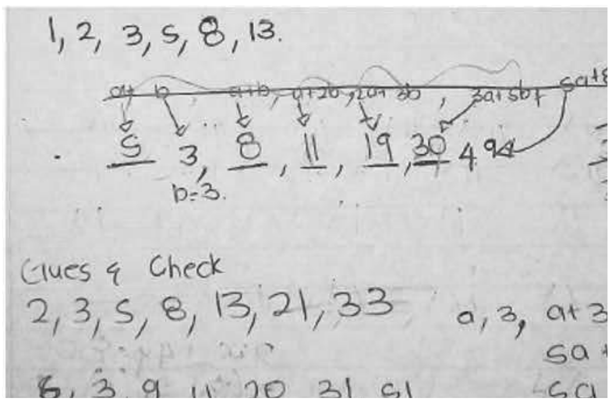


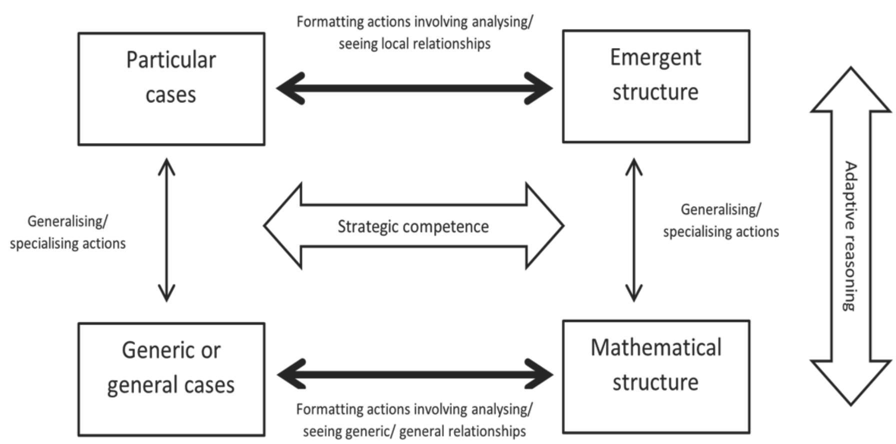


This PT starts with an *algebraic strategy* (code S_BA) as shown by the bold arrow working to the right) working with an unknown value x and creating the general structure of the required pattern sequence. However, the student constructs an incorrect equation $15 + 3x = 49$, possibly taking 15 and $3x$ to be two separate terms, by making the value of $T_6 = T_7$ instead of making $T_5 + T_6 = 49$. Unaware of her error, she calculates a value for x, but does not check her answer, either by revisiting (bold arrow to the left) the generic cases she set up, or by specifying (substituting) across the complete mathematical structure (Code A_0, for no adaptation). Her adaptive reasoning thus falls short on two occasions. In terms of the structure framework (on the right), the student starts with the general case (using x) and builds the mathematical structure correctly, but she does not check the emergent structure of the sequence.

Solution: PT 2

The second PT begins by working from the given pattern example of 1; 2; 3; 5; 8; 13 to set up a general *algebraic structure* in terms a and b (code S_BA). Based on his arrows and the colour of ink used, it is proposed that he then assigns $b = 3$ and the last term 49. Like PT 1 he sets up an equation for the last term in order to solve for a but uses the wrong terms. He strikes through the work and reverts to a deliberate guess and check strategy, working with particular cases towards *emergent structure* (code S_TA). He chooses different starting number values (2; 6 and 5) to find a Fibonacci pattern that ends with 49. He then returns to the task to use algebra and provides an alternative mathematical structure, using only one unknown a as the first term and builds the sequence according to the Fibonacci structure, making the last term $5a + 14 = 49$ (code A_EM). This equation is then used to solve for $a = 5$ (code S_BA). He ticks the sequence starting with 5 to show he verifies the sequence and fills in the missing numbers in the sequence at the top (code A_GP).

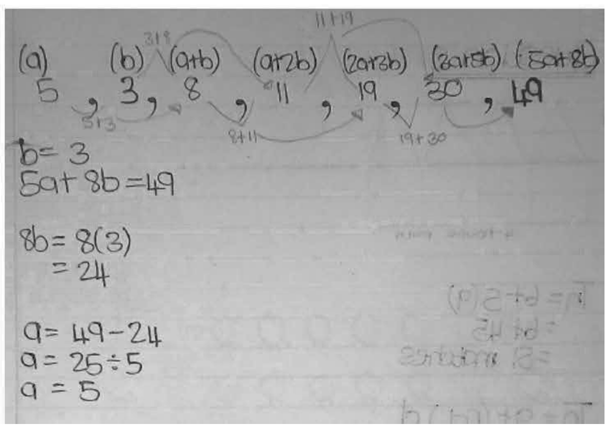


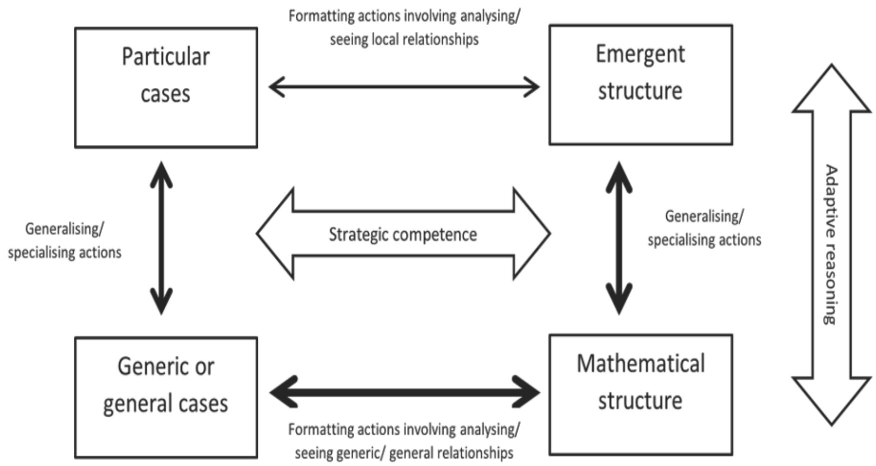


The deliberate use of both kinds of formatting actions to approach the problem shows strategic flexibility. This student demonstrates effective adaptive reasoning by moving to formatting emergent structure when his first mathematical structure was incorrect, then back to an appropriate algebraic structure, and back again to the emergent structure to verify the algebraic solution. In terms of the structure framework, the student was able to format both emergent and algebraic structure and to relate the two kinds of structures as well as the two kinds of cases by generalising and specialising actions to adapt his reasoning at key moments.

Solution: PT 3

The third test solution shows the PT starting with an algebraic strategy by making the first term *a* and the second term *b* to build an algebraic sequence with the Fibonacci structure (formatting on the bottom axis) (code S_BA). She then sets up the equation $5a + 8a = 49$, having matched the last terms of the algebraic sequence with the last term in the numerical sequence. We infer that at this stage the student had not yet determined the missing numerical terms.





The student substitutes $b = 3$ into the equation to transform $5a + 8b = 49$ into $5a + 24 = 49$. This is a 'convenient' specialising action (code A_GP) to eliminate the second variable in the algebraic structure of the sequence, so that she can solve for a. The student uses the structure of the equation $5a + 8b = 49$ but solves for a with the particular values of $5a$ and $8b$ in the equation, once again making convenient adaptations to support her strategy (Code S_BA and A_GP). To solve for a she creates a formula with $5a$ as the subject (we accept that she made a notation error by writing $a = 49 - 25$). It is likely that she then worked out the numerical sequence written at the top and showed with the additions in pencil that the numeric terms obey the structure of the Fibonacci sequence (code A_ME). The solution shows that the student is able to set up general and algebraic structure but seems more at home with local numerical calculations than algebraic manipulation. The interaction between the two formats keeps her on the right track and constitutes adaptive reasoning.

DISCUSSION

The analysis of the test question supports the view that structuring is multi-leveled rather than dichotomous (Antonides & Battista, 2022) when the purpose of the structuring is considered. This study showed that the PTs' strategic competence has the nature of seeing either local relationships, or general relationships; and adaptive reasoning has the nature of switching between general and emergent structures, or particular and general cases. While the test item that was analysed asked the PTs to use algebra to form the target sequence, the solutions revealed that the students “worked towards” an algebraic solution in different ways. In the case of PT 1, immediate and exclusive algebraic formatting obscured errors, which indicates that such students may benefit from explicit instruction about the different kinds of structures that are available for adaptive reasoning. PT 2 and PT 3 both showed awareness of the value of working with the emergent structure when they were unsure of the mathematical structure or its manipulation, and to recreate the mathematical structure from their actions on the numbers. Working on both levels with strategic and adaptive reasoning on purpose allowed the PTs to overcome insufficient strategies (Star & Rittle-Johnson, 2008). We propose that problem-solving courses for primary school PTs must make explicit that strategic thinking includes deliberate decisions about formatting actions toward emerging or mathematical structures, and that switching between these structures is a critical adaptive action that they must develop in their future learners.

The analytical framework that we developed is an initial step to analyse PTs' strategic competence and adaptive reasoning and addresses a gap in the literature on assessment of mathematical proficiency. Further research should explore the use of our analytical framework with a variety of test items that require PTs to communicate their reasoning.

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