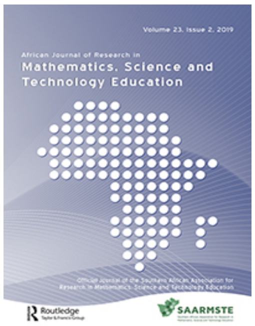


African Journal of Research in Mathematics, Science and Technology Education

ISSN: 1811-7295 (Print) 2469-7656 (Online) Journal homepage: https://www.tandfonline.com/loi/rmse20

Children's Capacity for Algebraic Thinking in the Early Grades

Dominique Afonso & Sharon Mc Auliffe

To cite this article: Dominique Afonso & Sharon Mc Auliffe (2019) Children's Capacity for Algebraic Thinking in the Early Grades, African Journal of Research in Mathematics, Science and Technology Education, 23:2, 219-232, DOI: 10.1080/18117295.2019.1661661

To link to this article: https://doi.org/10.1080/18117295.2019.1661661

Published online: 18 Oct 2019.

African Journal of Research in Mathematics, Science and Technology Education, 2019
Vol. 23, No. 2, 219–232, https://doi.org/10.1080/18117295.2019.1661661

Children's Capacity for Algebraic Thinking in the Early Grades

Dominique Afonso, and Sharon Mc Auliffe *

Faculty of Education, Cape Peninsula University of Technology, Cape Town, South Africa
**Corresponding author. Email: mcauliffes@cput.ac.za*

Patterns pervade our everyday life and are natural and familiar to children. Exciting new research in the field of Early Algebra describes the development of young children's algebraic thinking through the meaningful study of patterns. This paper contributes to this research by reporting on a case study of two primary schools in South Africa and looking at the capacity of learners for algebraic thinking in the early grades. Data was gathered using Grade 3 learner tests and focus group interviews based on pattern tasks. The data analysis process examined learners' pattern matching and building explanations related to each data collection instrument using Blanton, Brizuela, Gardiner, Sawrey and Newman-Owens's (2015. A learning trajectory in 6-year-old's thinking about generalising functional relationships. *Journal for Research in Mathematics Education*, *46*, 511–558) levels of sophistication in learner's thinking about functional relationships. The results of the study suggest that young learners have the potential to think algebraically when offered opportunities to do so, particularly in terms of recursive and functional thinking when solving pattern problems. Traditionally instructional practice emphasises lengthy exposure to recursive thinking as a first route to algebraic thinking: this recent research suggests that young children also have the capacity to engage in functional thinking. Thus, we may need to rethink our approach to algebraic thinking through the development of a pattern-based approach to algebra.

Keywords: *Early algebra; functional thinking; patterns; early grades; mathematics learners*

Introduction

Research into the teaching and learning of algebra is a well-established field, with good reason given that algebra is perceived to be the gateway to academic and economic success (Moses & Cobb, 2001). Recently, there has been growing interest in primary school algebra and the experiences needed for 'building informal notions about algebraic concepts and practices into formal ways of mathematical thinking' (Blanton et al., 2018, p. 29). While algebra continues to make up a large portion of high school mathematics, this new research shows that young learners, when provided with the opportunity, have the potential to think algebraically, suggesting that algebraic thinking needs to be developed over time and start in the early grades (Carpenter, Franke, & Levi, 2003). Blanton and Kaput (2004) propose that children need to learn a way of doing mathematics that is different from that of their parents: to be taught mathematics that extends further than arithmetic and computational fluency and includes opportunities to build, express and justify mathematical generalisations. Vermeulen (2008) supports this approach and recommends that algebraic thinking (Early Algebra, EA) should be developed in the primary school before the learner is exposed to algebra in high school. It is important to emphasise that EA is not about introducing algebra procedures and skills earlier but rather about developing ways of thinking and reasoning that benefit all aspects of mathematics (McAuliffe, 2013). Most EA research focuses on two aspects of algebra: *generalised arithmetic* and *functional thinking*. Here we focus on functional thinking as this provides a rich context for developing algebraic thinking practices involving generalising and reasoning, representing and justifying functional relationships (Stephens, Ellis, Blanton, & Brizuela, 2017). Research on functional thinking suggests that even young children can engage in this way of thinking and are able to represent how two different quantities correspond, using natural language, symbols and diagrams. Whilst the use of pattern tasks to develop functional thinking is widely reported in early algebra literature (Blanton et al., 2015; Carraher, Schliemann & Schwartz, 2008; Cooper & Warren, 2011), within the South African context, the algebraic potential of patterns tasks is fairly unknown, to both teachers and children, and is an under-researched topic. This paper draws from research investigating the development of algebraic thinking in the Foundation Phase using two different curricula: South Africa National Curriculum Standards (SA DBE, 2011) and Singapore Maths (SM; Singapore-ME, 2012). We focus on the children's capacity for algebraic thinking through the analysis of their solutions to pattern tasks at the Grade 3 level to investigate what children's solutions to pattern tasks tell us about their levels of understanding of functional relationships.

Literature Review

While there is some agreement that EA extends beyond arithmetic and computational fluency, different interpretations have emerged over the past 20–30 years as a result of on-going research. For some, EA involves the development of ways of thinking and reasoning such as analysing relationships, noticing structure, studying change, predicating, justify and proving (Kieran, 2004). For others, it is seen as a move away from working with numbers and measurements towards seeking relationships between sets of numbers and measurements (Carraher, Schliemann, Brizuela, & Earnest, 2006). The pioneering work of Kaput (2008) has helped to establish a deeper understanding of the core aspects and content strands involved in EA, including using symbols to generalise and acting on the symbols in the context of generalised arithmetic, functional thinking and modelling. We focus here on functional thinking as a way of helping children to see and describe mathematical structures and relationships and so construct meaning (Blanton, 2008). Teachers and learners working with patterns, looking for relationships, conjecturing and generalising these relationships using words and symbols, where appropriate, provides an access route into functional thinking (Blanton, 2008). The aim of functional thinking is thus to deepen children's conceptual understanding of mathematics and to recognise the generality within mathematics.

Pattern activities are a powerful tool in helping learners to develop functional thinking through providing concrete and transparent ways to describe, identify and analyse regularities and change, to make predictions and solve problems (SA DBE, 2011). They also provide opportunities to integrate algebraic thinking into an existing mathematics curriculum. Lannin (2004) describes two ways of reasoning that can emerge through learners working with pattern tasks: recursive and explicit. Recursive reasoning involves examining mathematical relationships between consecutive terms in a sequence in order to find the relationship between them and to use this relationship to find the next term in the sequence. Explicit reasoning implies children can not only 'determine the change from one instant to the other' but can also 'think about a general relationship that exists across many cases' (p. 219). When children are posed with a pattern problem, their natural instinct is to use recursive reasoning, which may not assist in developing the child's understanding of patterns and the relationships that exist between two or more varying quantities. Blanton and Kaput (2011) discovered that, when given ample opportunities to work with patterns, children start by using their natural language and through experiences and pedagogical support can move towards a more symbolic representation. Whilst some research on algebraic thinking highlights the challenges of shifting children's thinking from recursive to a more explicit functional perspective (Warren & Cooper, 2008), Blanton et al.'s (2015) work with first-grade children (6 years olds) on functional thinking showed that they were able to generalise functional relationships on different levels which could become more sophisticated through well-designed instruction.

While patterns tasks provide an obvious route to functional thinking, they are not without difficulty. Some research findings show that learners find it easier to extend patterns than to generalise (Zazkis & Liljedahl, 2002), and focus on the difference between consecutive terms when trying to generalise (i.e. recursive thinking) while ignoring the underlying structure needed to generalise across specific instances (i.e. functional thinking) (Noss, Healy, & Hoyles, 1997). This suggests that learners are more inclined to use pattern spotting and guessing when trying to generalise rather than look for a relationship between an element of the pattern and its position (Warren, 2005). There is also evidence that limiting pattern finding to recursive relations is less powerful in helping to develop learners' functional thinking.

Given the research findings of young children and their capacity for algebraic reasoning, it is important to highlight the role of the teacher in creating learning opportunities for this thinking. Barbosa and Vale (2015, p. 62) suggest that the design and use of tasks are important factors when developing algebraic thinking, requiring the teacher to 'reflect on the structuring and implementation of tasks to better promote the development of functional reasoning'. Teachers need to be aware of spontaneous opportunities arising in the classroom to encourage learners to think algebraically and make use of these to develop a different way of thinking about numbers and relationships (Hunter, Anthony & Burghes 2018). If these opportunities and activities are attended to in a meaningful way then, as Carraher et al. (2006) suggest, they have the potential to bring together mathematical ideas that may have otherwise remained isolated.

Theoretical Framework

The theoretical and subsequent analytical framework for this study is based on the Blanton et al. (2015) research with first-grade (6 years old) learners which characterises different levels of sophistication in thinking about functional relationships. These levels provide fine-grained elaboration of recursive and functional thinking and challenge 'the typical curricula approach' in which 'children consider only variation in a single sequence of values' (p. 511). Their findings suggest that, given exposure to appropriate tasks, even young children can learn to think about functions in more sophisticated ways and they do not need to advance through each level in a linear fashion in order to proceed to the next. We selected this framework for analysis of learner written and verbal responses as we wanted to know if it was applicable for use in the higher grades. Blanton et al. (2015, pp. 525–539) characterise eight levels of sophistication in children's thinking about functional relationships. Level 1 is titled *pre-structural*, meaning that the child does not describe or use any mathematical relationship in talking about problem data. In the next two levels (2 and 3), *recursive-particular* and *recursive-general*, the child conceptualises a recursive pattern either as a sequence of particular instances (e.g. counts, 2, 4, 6, …) or as a generalised rule between successive values ('add 2 every time'). The following three categories focus on different levels of functional relationships: Level 4, *function-particular*, means that a learner can 'describe a relationship within specific cases but not as a generalised functional relationship' (e.g. $6 + 1 = 7$; $7 + 1 = 8$ and $8 + 1 = 9$); Level 5, *primitive functional-general*, means that a learner can recognise a functional relationship but is not able to articulate a relationship to 'identify the specific parts being compared in the generalised form' (e.g. when given a rule, learners can evaluate its correctness but cannot produce the rule themselves); and Level 6, *emergent functional-general*, means that they can 'identify quantities being compared when describing the functional relationship but … not specify the transformation between the quantities' (e.g. I double the number of stops to get the number of cars). Level 7, *condensed functional-general*, involves understanding that two quantities are related and how to represent the transformation between these quantities as a rule (e.g. I double the number of stops to get the number of cars is given as the function $2S = C$). Level 8, *function as object*, means that learners seem to understand that a relationship is held across a set of values but recognise that it is no longer valid if the problem situation changes.

Methodology

This research was part of a larger study investigating how two South African schools, using different mathematics curricula, created opportunities to develop children's algebraic thinking. The study was conducted in two Grade 3 classes, at two primary schools in Cape Town, one using the South

African curriculum guidelines (CAPS) and the other, the Singapore mathematics curriculum. Both are private schools and classified quintile 5 under the government quintile rating system, i.e. relative wealth of surrounding communities. Each draws learners from middle to higher income sectors and the schools are considered to be well resourced. The sample consisted of all learners and their teachers from one Grade 3 class at each school. One class had 28 learners. the other 24 learners. Data was collected using a variety of methods including document analysis, a pencil and paper early algebra test, teacher interviews and focus group learner interviews. In this article, we focus on what children's solutions to pattern problems tell us about their levels of understanding of functional relationships, using data from written items and focus group interviews.

Learners from both schools were given an early algebra written test (Appendix 1), including generalised arithmetic and function thinking items, based on content related to number and geometric patterns. The selection of items was guided by the curricula and textbooks used in classroom and Annual National Assessment items (SA-DBE, 2013). We included two contextual (quasi-real and figurate) pattern problems: Questions 7 and 8 to focus more specifically on learner's functional thinking. Learners completed the test individually under the supervision of their teachers. The tests were graded for correct and incorrect answers and the results summarised to give an overall performance for each of the classes. All items related to justification were then qualitatively analysed using the Blanton et al. (2015) levels in understanding of functional relationships. The quantitative test results were tabulated into the different groups (South Africa and Singapore Maths) to give the overall performance of the cohort (and not to compare the different curricula which is outside the scope of this paper).

Written Test Design

The test comprised nine items covering content topics related to algebraic thinking, classified into three categories: number counting patterns, mathematical equivalence and functional thinking. Category one was included given the importance of using number pattern problems in the younger grades to focus on how quantities vary and relate to each other (Blanton & Kaput, 2004). Only when this is achieved, as opposed to simple patterning, does the thinking involved become a tool to assist in the development of algebraic thinking required in later grades. The second category of items, equivalence, was included as working with the concepts of equality and variables are seen as critical foundations for further algebra learning yet are often poorly understood at the higher grades with the equal sign interpreted as an instruction to operate with numbers rather than understanding that the values on either side of the sign are equivalent (Carpenter et al., 2003).

The final category included three items focused on learners' understanding of functional thinking using different representations. The first item related to functional thinking and elementary patterning using a quasi-real-life context requiring the learner to recognise–continue–describe–explain regularity within the pattern (Question 7). The ability to do this is recognised as a pathway to generalising and the idea of functional relations (Steinweg, Akinwunmi, & Lenz, 2018). The second was a figural pattern task (Question 8), which is perceived as a useful way to encourage learners to describe patterns and make generalisations about them (Charles & Carmel, 2005). Such tasks are not common in Foundation Phase textbooks for either CAPS or SM but this item was included to understand children's capacity for functional thinking given similar research in other countries (Blanton et al., 2015; Carraher et al., 2006; Radford, 2018) The third task involved a function (flow) diagram with a given function rule and required learners to calculate unknown input and output values. This is a common task in Grade 3 textbooks in both curricula. All three tasks were included to explore children's understanding of functional thinking and their use of and flexible shifting between different representations, which is seen as an indicator of an understanding of functions and relationships (Blanton, 2008).

Focus Group Interviews

There were two group interviews, one from each class, each group comprising eight learners of mixed abilities (teacher chosen based on class results). Participants were asked to discuss and solve two different pattern tasks (contextual and figural—Appendix 2), similar to those used in the written test for Questions 7 and 8. The aim was to probe more deeply, using the Blanton et al. (2015) levels, into learners' thinking when solving pattern problems. These types of problems were selected as they are frequently identified in literature as the type of tasks that encourage functional thinking and building relationships. While the problems involved similar number patterns to those recommended in both curricula, the context and format were different from those found in textbooks at this grade level. The first pattern task used a context familiar to the learners, i.e. dogs and eyes, in helping them to visualise the story ($y = ax$), while the second task provided the learners with the opportunity to physically build the first few terms in order to gain a deeper understanding of the structure of the pattern problem ($y = ax + b$) (Blanton et al., 2015).

The focus groups interviews were audio-recorded while learners worked on the tasks with the researcher facilitating the discussion. The patterns tasks were completed one after the other to give opportunity for group work and discussion. Learners were encouraged to work in pairs and to explain their strategies. The researcher joined the pair discussions, asking probing questions to encourage learners to justify their solutions. The focus group interview recordings and responses were transcribed, coded and analysed using the Blanton levels.

Findings

The findings are presented in three sections. Firstly the overall performance of both classes is given based on learner quantitative answers only, providing a summary of the two groups in relation to generalised arithmetic and functional thinking. The second category relates to learner responses to Question 7 and 8 as these best relate to functional thinking and the third section relates to the focus group interviews. In these latter two sections, our intention is not to provide an exhaustive analysis but to show the potential for functional thinking.

Written Test Results

The results of the written test (Table 1) indicate that children are more successful with number counting patterns, as might be expected given the typical tasks used in Grade 3 in both schools, and least confident when working with problems related to equivalent equations and the function table (Question 7). The poor learner achievement in some aspects of equivalence is not dissimilar to those found in international studies, which suggest that difficulties with equivalence are linked to limiting experiences with arithmetic and inappropriate generalisation (McNeil, 2014). Pattern problems such as those given in Question 7 and 8 are not typically found in either the CAPS or the SM systems but show the potential of young children to engage in functional thinking. Differences in the results for each group may be associated with the different curricula and resources used in the classroom but this is not the focus of this article, although it is interesting to note that the CAPS school's results are generally higher.

Illustrative Learner Responses to Specific Questions

This section focuses more specifically on findings related to Questions 7 and 8 in both the written test responses and in the focus group interviews given that these types of tasks help create the potential

Table 1. Percentage performance on individual test items

Question and content	Singapore Maths School, percentage correct	CAPS school, percentage correct
Q1: Number counting pattern	84	96
Q2: Number counting pattern	88	86
Q3: Number counting pattern	64	93
Q4: Equivalence	60	75
Q5: Equivalence	32	44
Q6: Finding an unknown	56	75
Q7: Function table	36	41
Q8: Geometric (figural) pattern	48	68
Q9: Flow (function) diagram	40	86

and opportunity for learners to generalise 'relations between co-varying quantities, express those relationships in words and symbols and reason with these representations to analyse functional behaviour' (Blanton, Levi, Crites, & Dougherty, 2011, p. 47). While the overall computational results of the test reveal low levels of success for Question 7, we were interested in the variety of written explanations given by learners and what they reveal about levels of functional thinking. We acknowledge that learner writing may not capture the full extent of learner understanding but it does help us to understand the potential of learner thinking when solving pattern tasks. We analysed all learner computational and justification answers for functional thinking and selected examples of their responses to show different levels of thinking in their written explanations.

The first learner solution (Figure 1) shows a vertical pattern difference between the individual input and output values as multiples of 2 and the use of this to calculate all missing values. Lack of attention to the change in or position of values results in the difference being incorrectly added to output value 30 to calculate a corresponding input value. The answer for Question 7a shows that the learner divided 150 by 2 to find the number of days for the flower to grow to 150 cm. The reasoning given, 'I added the numbers', appears to make reference to the table without any further clarification. This written solution exhibits thinking at the *recursive-particular* level (Level 2) as the learner appears to conceptualise the pattern (incorrectly) as a sequence of particular instances (+2, +4, +6) and not as a general relationship.

The next learner solution (Figure 2) shows the correct answers for the missing input and output table values and the number of days for a height of 150 cm. The learner justification reflects thinking at *emergent-functional general* (Level 6) by identifying the mathematical transformation between the quantities as a functional relationship: 'I used the 3 and 10 times tables to work it out' and includes examples. The learner does not yet exhibit *condensed functional-general* thinking (Level 7) which requires explicit conceptualisation of functions as a generalised relationship through both words and variable notation (Blanton et al., 2011, p. 535).

The next set of learner solutions relate to the figural pattern problem and demonstrate two different ways in which the learners used the spatial composition of the problem to make sense of the pattern problem, to look for a relationship and to express this relationship in numbers and/or symbols (Blanton et al., 2011). The first learner (Figure 3) records the figural pattern as groups of squares with four matches and ignores the overlapping match in pictures 2 and 3. The square pattern is continued and used to draw pictures 4 and 6, and to calculate the number of matches needed for picture 20. While the pattern is continued incorrectly, the explanation reveals a link between the spatial pattern

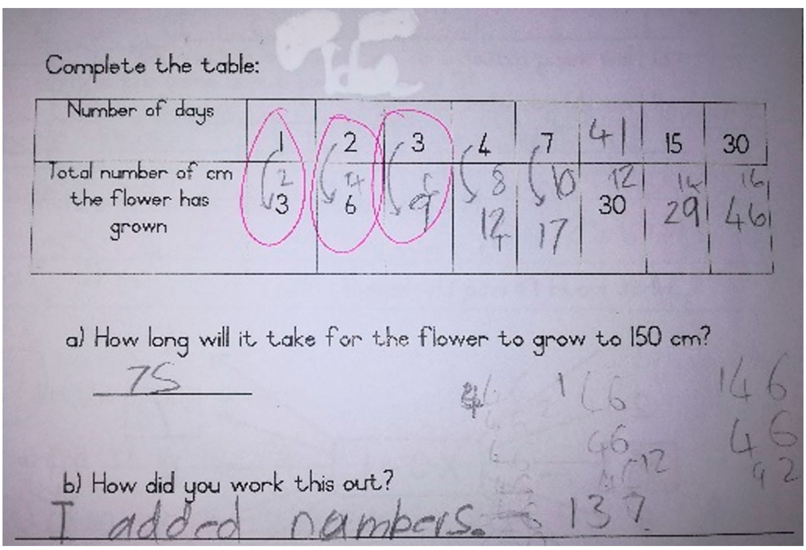


Figure 1. Example answer from Question 7

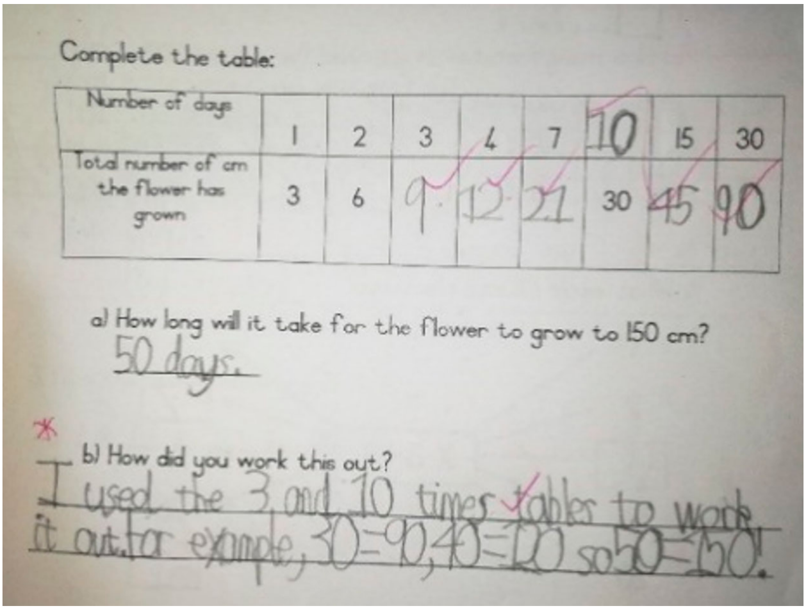


Figure 2. Example answer from Question 7

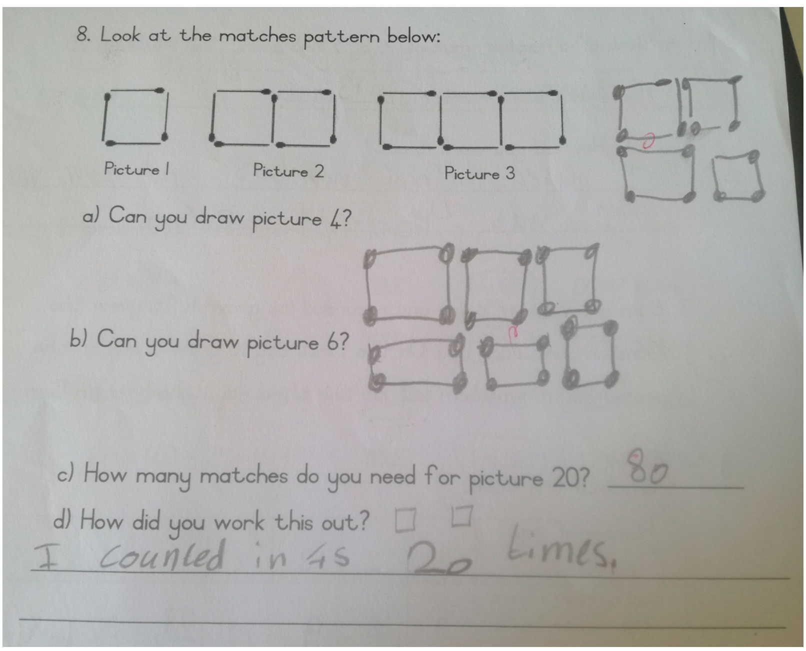


Figure 3. Example answer from Question 8

and the numerical expression. The learner thus exhibits *recursive general*-thinking (Level 3) as they conceptualise the pattern recursively and provide a general rule for 'this' pattern: 'I counted in 4s 20 times'. Given the emphasis on counting, this is not yet a 'generalised functional relationship over a class of instances' (Level 4, *functional-particular*) (Blanton et al., 2011, p. 530).

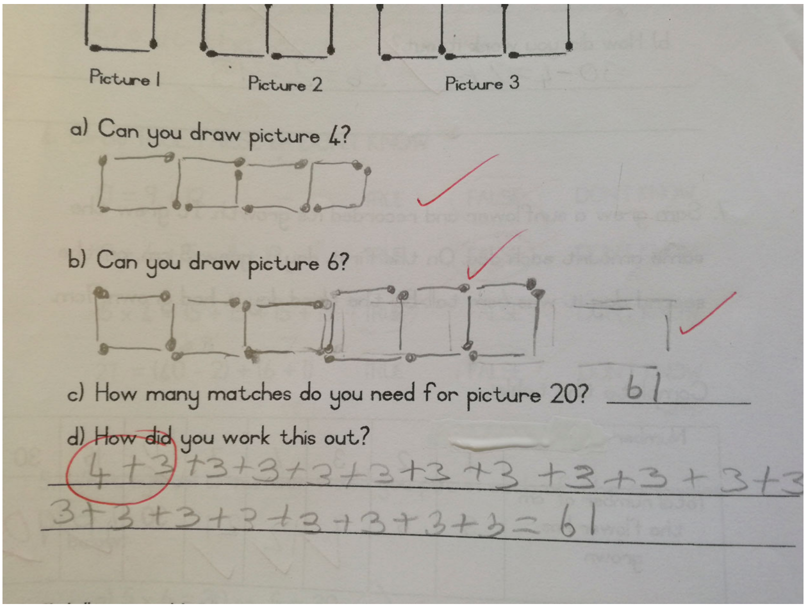


Figure 4. Example answer from Question 8

The leaner whose work is shown in Figure 4 calculates a correct solution to the figural pattern problem exhibiting signs of *emergent functional-general* (Level 6) thinking through the use of visual and written explanations. The drawings show the addition of three matches each time and the corresponding explanation in (d) identifies the solution for picture 20 as 4 + (3) nineteen times. The learner explains the relationship specific to this case using numbers but does not yet represent the relationship using general terms. This solution reflects thinking beyond *primitive functional-general* (Level 5) because the learner can not only describe a general relationship using numbers but is also able to articulate the relationship between the 'specific quantities being compared in their generalised form' (Blanton et al., 2011, p. 532).

Focus Group Interviews

The focus group interviews were designed to further understand learners thinking, when solving quasi-real pattern problems (context and figural). The first problem required learners to explain a relationship between the number of dogs and the number of eyes (Appendix 2: Question 1). They were encouraged to construct vertical data tables (Figure 5) to record their data and to make use of natural, spoken language (not symbols), which Blanton and Kaput (2011) explain is a normal starting point when solving the problem.

Both groups of learners exhibited both *recursive-particular* thinking (Level 2) to extend the dog and eyes pattern. When asked how many dogs there were if they saw 30 eyes, two learners impulsively shouted 60 dogs, then quickly changed their minds. One of the learners counted on her fingers and calculated the correct answer. When asked how she got her answer, she explained:

> I know that there was 30 eyes so I counted in twos up till 30 and then you would see how much dogs you would get.

In the triangles and matches task (Appendix 2—Question 2), one learner explained the rule as 'you start off with 3 and always plus 2'. By using the word '*always*', the learner exhibits understanding of a generalised rule that can occur between values (*recursive-general*—Level 3) but does not as yet use this to predict an answer for the number of matches needed for any number of triangles.

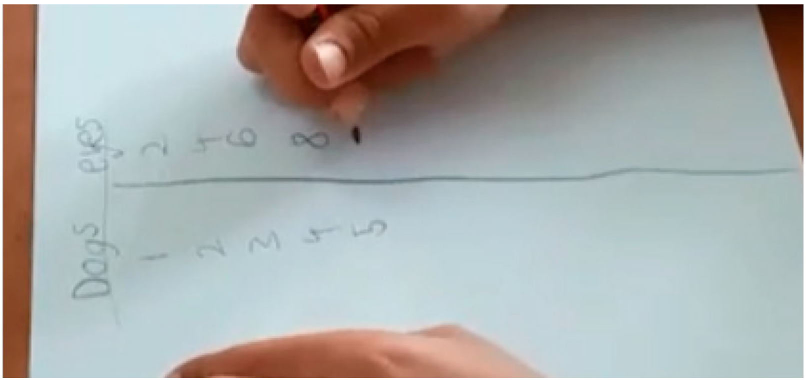


Figure 5. Example of a data table used in the dog and eyes task

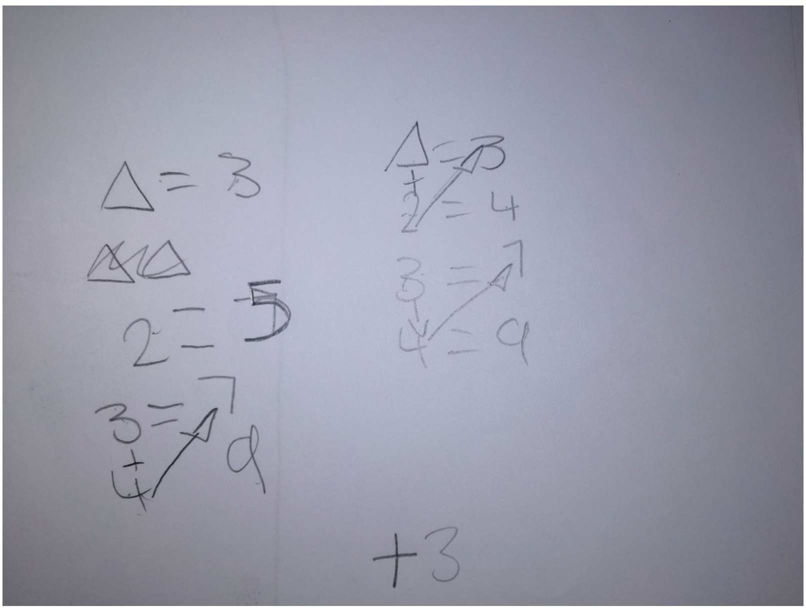


Figure 6. Learner's solution to triangles and sticks question

From the learner working shown in Figure 6, we see the learner exploring different ways to find a relationship between the number of triangles and the number of sticks needed to build them. The arrows indicate that she saw a pattern by adding consecutive input values to calculate a corresponding output. After working through the problem for some time, she finally explained her findings as follows:

Interviewer Can you tell me what you found?
Learner 3 Yes, well it will always be double and then minus 1.
Interviewer Why minus 1?
Learner 3 Because you had 1 (stick) in the beginning.

Using experimentation with numbers and the drawing, the learner exhibits *emergent functional-general* thinking (Level 6) through 'the emergence of key attributes of a generalised relationship although the representation of the relationship is incomplete' (Blanton et al., 2015, p. 533). The learner has yet, however, to represent the functional relationship in both words and variable notation (*condensed functional-general*—Level 7).

Discussion and Conclusion

Blanton (2008) describes functional thinking in terms of building, expressing and justifying mathematical relationships, which can be developed using patterns tasks. Her later work looks more closely at the nature of young children's thinking about generalising functional relationships and suggests that 6-year-olds have the capacity to think in 'generalised ways about relationships in function data' (Blanton et al., 2015, p. 511). We were interested in looking at Grade 3 learner written and verbal solutions to patterns tasks and what they tell us about their levels in understanding of functional thinking. We found that Grade 3 learners can exhibit different levels of functional thinking across both quasi-real and figural pattern problems. There is some evidence of recursive-particular and recursive-general, which is not unexpected given the current approach to patterns in the Foundation Phase. However, we also see from the findings (Figures 2 and 4) that some Grade 3 learners have the capacity to exhibit emergent functional-general thinking to solve problems and to build mathematical generality. The current approach to algebra in the Foundation Phase is to support number concept development and operational sense and the application of knowledge of space and shape (SA DBE, 2011). While there is some mention in the patterns, functions and algebra strand of pattern regularities and change, predictions and solving problems, the reality is that there is too little emphasis on functional thinking in the early grades. Many pattern problems focus on difference between consecutive terms, leading to an over-emphasis on recursive thinking, as indicated in some of the learner solutions (Lannin, 2004). This can present challenges later in shifting learner thinking from recursive to functional thinking and result in pattern spotting and guessing as a preferred solution strategy and ignoring the algebraic structure (Warren & Cooper, 2008).

It is important to note that, while young children may have the capacity to reason about patterns and functions, and to develop and use different representational tools, this way of thinking does not happen spontaneously and requires well-designed activities with deliberate support and scaffolding by the teacher (Blanton & Kaput, 2011). When using pattern tasks, teachers need to exploit opportunities to develop the children's thinking about functional relationships, or to redesign tasks if these opportunities do not exist (Barbosa & Vale, 2015; Hunter, Anthony & Burghes 2018). Hewitt (2019) argues that it may be preferable to use pattern problems that make use of contextual situations to encourage learners to look for structure, as he cautions that teaching practices often turn attention away from structural (algebraic) thinking towards arithmetic thinking. This happens when teachers using patterns tasks focus learners' attention towards the numbers only in a table of values. There is some evidence of this thinking in the learner written responses (Figures 1 and 6) in which they focus on the number relationship within and between terms, but not on generality. A figural approach may be more effective than a numerical reasoning approach for generalising patterns since 'it involves recognising a relationship between the parts of the figures forming the pattern and the picture number' (El Mouhayer & Jurdak, 2016, p. 212). The use of different contexts to build functional thinking can present challenges and affordances for learners. While tables are familiar representations used to develop functional thinking, they may hamper access to generality and result in learner over-use of recursive thinking. In contrast, figural pattern problems can present the opportunity for learners to make sense of the structure of the geometric drawing and to look for relationships (Hewitt, 2019).

In conclusion, the results of this small-scale study show that learners in the early grades have the capacity to think algebraically and functionally when presented with patterns tasks that require generality. Although the sample size is limited and the selection of the schools is narrow, it encourages us, when using pattern tasks, to move beyond arithmetic thinking with a focus on process and numerical results towards developing functional thinking. We acknowledge that there are different approaches to help learners make sense, represent and reason about functional relationships but we also recognise the urgent need to provide children with 'sustained interactions' in which 'algebraic habits of mind can be forged' to build a foundation for the formal study of algebra later (Blanton et al., 2015, p. 545). This type of research highlights the need for prolonged and further study of the use of a pattern-based approach to develop algebraic thinking in the early grades.

Disclosure Statement

No potential conflict of interest was reported by the authors.

ORCID

Sharon Mc Auliffe http://orcid.org/0000-0003-2777-9504

References

Barbosa, A., & Vale, I. (2015). Visualisation in pattern generalisation: Potential and challenges. *Journal of the European Teacher Education Network*, *10*, 57–70.
Blanton, M. (2008). *Algebra and the elementary classroom*. Portsmouth: Heinemann.
Blanton, M., Brizuela, B. M., Gardiner, A. M., Sawrey, K., & Newman-Owens, A. (2015). A learning trajectory in 6-year-old's thinking about generalizing functional relationships. *Journal for Research in Mathematics Education*, *46*, 511–558.
Blanton, M., Brizuela, B. M., Stephens, A., Knuth, E., Isler, I., Gardiner, A. M., … Stylianou, D. (2018). Implementing a framework for early algebra. In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5- to 12- year olds* (pp. 27–49). Cham: Springer.
Blanton, M., & Kaput, J. (2011). Functional thinking as a route into algebra in the elementary grades. *International Reviews on Mathematics Education*, *37*, 34–42.
Blanton, M. L., & Kaput, J. J. (2004). Functional thinking as a route into algebra in the elementary grades. In J. Cai, & E. Knuth (Eds.), *Early algebraization. Advances in mathematics education* (pp. 5–23). Berlin: Springer.
Blanton, M., Levi, L., Crites, T., & Dougherty, B. (2011). *Developing essential understandings of algebraic thinking, grades 3–5*. Reston, VA: National Council of Teachers of Mathematics.
Carpenter, T., Franke, M., & Levi, L. (2003). *Thinking mathematically*. Portsmouth: Heinemann.
Carraher, D. W., Schliemann, A. D., Brizuela, B. M., & Earnest, D. (2006). Arithmetic and algebra in early mathematics education. *Journal for Research in Mathematics Education*, *37*(2), 87–115.
Carraher, D. W., Schliemann, A. D., & Schwartz, J. L. (2008). Early algebra is not the same as algebra early. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 235–272). New York: Lawrence Erlbaum Associates.
Charles, R. I., & Carmel, C. A. (2005). Big ideas and understandings as the foundation for elementary and middle school mathematics. *Journal of Mathematics Education Leadership*, *8*, 9–24.
Cooper, T. J., & Warren, E. (2011). Years 2 to 6 students' ability to generalise: Models, representations, and theory of teaching and learning. In J. Cai, & E. Knuth (Eds.), *Early algebraization* (pp. 187–214). New York: Springer.
El Mouhayer, R., & Jurdak, M. (2016). Variation of student numerical and figural reasoning approaches by pattern generalisation type, strategy use and grade level. *International Journal of Mathematical Education in Science and Technology*, *47*(2), 197–215.
Hewitt, D. (2019/in print). "Never carry out any arithmetic": The importance of structure in developing algebraic thinking. In U. T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the eleventh congress of the European society for research in mathematics education*. Utrecht, the Netherlands: Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.
Hunter, J., Anthony, G., Burghes, D. (2018). Scaffolding teacher practice to develop early algebraic reasoning. In C. Kieran (Eds.), *Teaching and Learning Algebraic Thinking with 5–12 year olds* (pp. 379–401). ICME-13 Monographs. Springer, Cham. http://doi.org/10.1007/978-3-319-68351-5_2.
Kaput, J. J. (2008). What is algebra? What is algebraic reasoning? In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 5–17). New York: Lawrence Erlbaum Associates.
Kieran, C. (2004). Algebraic thinking in the early grades: What is it? *The Mathematics Educator*, *8*(1), 139–151.
Lannin, J. K. (2004). Developing mathematical power by using explicit and recursive reasoning. *Mathematics Teacher*, *98*(4), 216–223.
McAuliffe, S. (2013). *The development of preservice teachers' content knowledge for teaching early algebra* (Unpublished doctoral dissertation). Cape Peninsula University of Technology, Cape Town.

McNeil, N. M. (2014). A change-resistance account of children's difficulties understanding mathematical equivalence. *Child Development Perspectives*, *1*(1), 42–47.
Moses, R. P., & Cobb Jr, C. (2001). *Radical equations: Civil rights from Mississippi to the algebra project*. Boston, MA: Beacon Press.
Noss, R., Healy, L., & Hoyles, C. (1997). The construction of mathematcal meanings: Connecting the visual with the symbolic. *Educational Studies in Mathematics*, *33*(2), 202–233.
Radford, L. (2018). The emergence of symbolic algebraic thinking in primary school. In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5- to 12- year olds* (pp. 3–26). Cham: Springer.
Singapore. Ministry of Education. (2012). *Mathematics syllabus primary*. Retrieved from https://www.moe.gov.sg/docs/defaultsource/document/education/syllabuses/sciences/files/primary_mathematics_syllabus_pri1_to_pri5.pdf.
South Africa. Department of Basic Education. (2011). *Curriculum and policy assessment statement: Foundation phase*. Pretoria: Government Printing Works.
South Africa. Department of Basic Education. (2013). *Annual national assessment—Grade 4*. Pretoria: Government Printing Works.
Steinweg, A. S., Akinwunmi, K., & Lenz, D. (2018). Making implicit algebraic thinking explicit: Exploiting national characteristics of German approaches. In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5- to 12- year olds* (pp. 283–308). Cham: Springer.
Stephens, A. C., Ellis, A. B., Blanton, M., & Brizuela, B. M. (2017). Algebraic thinking in the elementary and middle grades. In J. Cai (Ed.), *Third handbook of research on mathematics teaching and learning* (pp. 386–420). Reston, VA: National Council of Teachers of Mathematics.
Vermeulen, N. (2008). *Algebraising the arithmetic curriculum*. Cape Town: Cape Peninsula University of Technology.
Warren, E. (2005). Young children's ability to generalise the pattern rule for growing patterns. In H. L. Chick, & J. L. Vincent (Eds.), *Proceedings of the 29th conference of the international group for the psychology of mathematics education* (Vol. 4, pp. 305–312). Melbourne: PME.
Warren, E., & Cooper, T. J. (2008). Patterns that support early algebraic thinking in the elementary school. In C. E. Greenes, & R. Rubenstein (Eds.), *Algebra and algebaric thinking in school mathematics* (pp. 113–126). Restn, VA: NCTM.
Zazkis, R., & Liljedahl, P. (2002). Generalisation of patterns: The tension between algebaric thinking and algebraic notation. *Educational Studies in Mathematics*, *49*, 379–402.

Appendices

Appendix 1: Pencil and Paper Test

1. 545, _______, 535, 530, 525, _______, ________, ________
   Fill in the missing number. What do you notice?
   ____
   ____
2. 490, 494, _____, 502,______, 510, _______, ________
   Fill in the missing numbers. What do you notice?
   ____
   ____
   What is the same about Questions 1 and 2?
   ____
   ____
   What is different about Questions 1 and 2?
   ____
   ____
3. Fill in the missing numbers:
   (a) 4, 8, ____, 16; ____; ____; ____; ____; _____
   (b) 40, 80, ____, 160; ____; ____; ____; ____; _____

What pattern do you notice?
____

4. Circle TRUE, FALSE or DON'T KNOW

$19 = 9 + 12$	TRUE	FALSE	DON'T KNOW
$10 + 6 + 5 = 11 + 12$	TRUE	FALSE	DON'T KNOW
$15 \times 4 = 15 + 15 + 15 + 15$	TRUE	FALSE	DON'T KNOW
$27 = (40 \div 2) + (6 + 1)$	TRUE	FALSE	DON'T KNOW

5. What symbol would you put in each of the triangles below:
   (a) $5 \times 6 = 30$ then $5 = 30 \Delta 6$
   (b) $6 + 9 = \Delta + 10$
   (c) $18 + 9 + 16 = \Delta + 12$
   (d) $\Delta + 7 = 3 + 6 + 4$
6. Think of a number, multiply by 2 and add 4. The answer is 30
   (a) What is my number? ________
   (b) How did you work it out?
   ____
   ____
7. Sam grew a sunflower and recorded its growth. On the first day it grew 3 cm, on the second day it was 6 cm tall. By the third day it had grown 9 cm.

Complete the table:

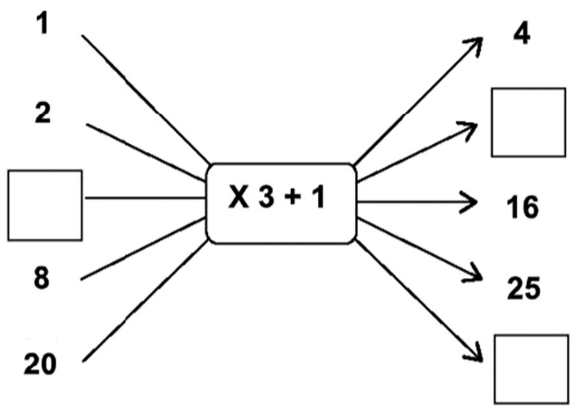
Number of days	1	2	3	4	7		15	30
Total number of cm the flower has grown	3	6				30		

(a) How long will it take for the flower to grow to 150 cm? ________________
(b) How did you work this out?
____
____

8. Look at the matches pattern below:

(a) Can you draw picture 4?
(b) Can you draw picture 6?
(c) How many matches do you need for picture 20? ________________________
(d) How did you work this out?
____
____

9. What would fill into the boxes?



Appendix 2: Focus Group Interview Questions

(1) Dogs and eyes:

I went to a park and saw lots of dogs. I wanted to know how many eyes there were altogether. Do you think you could help me work it out?
First I saw 2 dogs, how many eyes?
4 dogs?
6 dogs?
Then I saw 10 dogs. How can I work this out? Must I count every dog's eyes?
What if there were 20 dogs?

(1) Triangles and sticks:

Look at the triangles we have built using the sticks. With a friend, can you answer the following questions:

How many sticks do we need to make **3** triangles?
How many sticks do we need to make **5** triangles?
How many sticks do we need to make **10** triangles?
What if I wanted to make **20** triangles? Is there a rule we can use?