Engaging Pre-Service Teachers' Flawed Solution Strategies in Number Context to Develop Structure Sense

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ABSTRACT

This paper examines the solution strategies used by pre-service teachers[1] when asked to solve a number problem as a means to engage them with the notion of structure sense. I report on how pre-service teachers' flawed solution strategies provided opportunities for critical inquiry into the structure of the number context and engagement with the process of validating their solution strategies. I will show how the process of validating moves the pre-service teachers through higher levels of generalisation. In this paper I argue that the notion of developing structure sense within number contexts provides primary school pre-service teachers with a rich context in which to engage with structure sense and generalisation as interrelated mathematical processes.

Keyword: solution strategies; structure sense; critical inquiry; generalisation;

BACKGROUND TO THE STUDY

This case study is an exploration of primary school pre-service teachers' engagement with a number problem in which alternative solution strategies to the standard algorithm is used as an opportunity to develop the notion of structure sense. The purpose of this exploration is to explore how a number problem context develops structure sense when engaging students with errors in their solution strategies.

The Curriculum and Assessment Policy Statement (CAPS) defines number sense development as including: the meaning of different kinds of numbers; the relationship between different kinds of numbers; the relative size of numbers; the representation of numbers in various ways; the effect of operating with numbers and the ability to estimate and check solutions (National Curriculum Statement, Senior Phase Grades 7-9, p10). The definition of structure sense is not made explicit in the CAPS. The development of number sense within CAPS

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[1] In this paper the pre-service teachers will be referred to as the students.

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emphasises the representation of numbers in different ways which includes the recognition of equivalent forms, for example, 3 + 4 and 2 + 2 + 3, which can be considered a move to focussing on the structural elements of the expressions.

A review of the literature (Wagner & Kieran, 1989; Booth, 1989; Linchevski & Herscovics, 1996; Linchevski & Livneh, 1996) suggests that structure sense requires learners to recognise that two number expressions have the same value because of common structural elements. For example, while 3+4 and 2+ 2 + 3 have different surface features, the structure of the number expressions has the form 3 + a, and a + 3, which represents the same number because of the commutative property of addition. Structure sense will demand that the learners explain why the two numbers expressions have the same value without having to compute the value of the expressions but instead focussing on the properties of the number system. The latter aspect of structure sense is not made explicit: CAPS focusses on using number properties to compute rather than to compare the structure of equivalent number expressions.

In this study pre-service teachers were introduced to these differences between number and structure sense to engage them with ideas of how these notions are related and to develop structure sense. The literature review in the next session provides a brief overview of some key research studies into the notion of structure sense in school mathematics.

LITERATURE REVIEW OF STRUCTURE SENSE

The notion of structure sense as it applies to school mathematics developed from research that explored the connections between arithmetic and algebra (Lee & Wheeler, 1989). Lincheski and Herscovics's (1996) research focussed on learners' solutions of equations and found that many of the errors in learners' solutions were linked to incorrect manipulation of the numerical parts of the equation. For example, the equation 4 + n – 2 + 5 = 11+ 3+ 5 was solved incorrectly because learners grouped the 2 and 5 on the left to produce 4 + n + 7 = 18. This error was referred to as the 'detachment of a term from the indicated operation' by Linchevski and Herscovics (1996**)** and they argue that this error is associated with learners' difficulty in uncovering the mathematical structure of the expression. Booth (1998) argues that learners' difficulties in algebra are in part due to their lack of understanding of structural notions of arithmetic.

Kirshner (1989) argues that for some learners the surface features and visual cues of expressions dominate their manipulation strategy rather than applying the correct mathematical properties. The research of Liebenberg, Linchevski, Sasman and Olivier (1999) suggest similar findings when learners performed calculations in a numerical context. For example, learners know that within the numerical expressions 14 + 2 x 6 and 14 x 5 + 5 that multiplication is the first order of operation and may even know why it is important for there to be order of operation. Yet there is a greater likelihood that learners will compromise this mathematical property in the second expression, 14 x 5 +5, adding 5 +5 = 10, and then multiplying.

Lüken's (2012) research focussed on structure sense of very young learners in their first year of school. The learners in Lüken's (2012) research were given geometrical patterns in which the pattern needed to be continued or to count the number of objects in a geometrical structure. The key finding of Lüken's (2012) research is that while all the learners could recognise the sub-structures in a pattern, the difficulty for some learners is recognising and establishing mutual connections and relations between sub-structures which support numerical procedures.

Kieran (1988) defined structural knowledge as the ability to identify equivalent forms of expressions. Linchevski and Vinner (1990) argued that this definition should be modified to include the ability to discriminate between the forms relevant to the task – generally one or two forms – and all the others. Hoch and Dreyfus (2004) define the notion of structure sense as it applies to high school algebra as a collection of the following abilities:

- To see an algebraic expression or sentence as an entity
- To recognise an algebraic expression or sentence as a previously met structure
- To divide an entity into sub structures
- To recognise mutual connections between structures
- To recognise which manipulations it is possible to perform
- To recognise which manipulations it is useful to perform

Hoch and Dreyfus (2004, p3.)

In this paper I will draw on the notions of structure sense as developed within the research described in this literature review to explore how structure sense can be developed in a numerical context to create opportunities for developing the thinking processes involved in generalisation.

METHODOLOGY

Participants

Prospective mathematics teachers at our institution[2] are streamed into the specific phases of the schooling system: foundation; intermediate – senior phase[3] and the further education and training phase. The students involved in this case study are first year and fourth year students who are in the intermediate-senior phase stream (ISP students). The fourth year students are a group of 17 students who specialize in mathematics. The first year students are students who did mathematics at school.[4] Most of the students are from the historically privileged group in South Africa and attended high schools that are referred to as ex-Model C[5] schools. The majority of the current prospective teachers at our institution followed the outcomes-based curriculum. Critical inquiry[6] was one of the core outcomes of the outcomes-based curriculum to develop critically-minded citizens for the new democracy.

Teaching Context

The mathematics education course for ISP students includes non-routine mathematics problems which students solve collaboratively and students are given the opportunity to present their solutions to the class. The non-routine problem that is the focus of this paper was given to the first and fourth year mathematics students in the first term of the academic year. The first year students had completed topics on number that included working with numbers in different bases; place value and number theory. The fourth year students completed topics in algebra that focussed on the solving of different kinds of equations. Some of the algebraic equations that were given to the fourth year students were taken from Hoch and Dreyfus's (2004) research[7] to see whether students' solution strategies would reflect a structural approach or not. The fourth year students in this study produced the same lack of structural sense as the students in Dreyfus's (2004) research.

The students were asked to complete the problem during the lecture period and to submit their solution to the problem at the end of the period. Written

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[2] A university of technology

[3] Intermediate-Senior Phase (ISP) includes grade 4 to 9. Grade 4 to 7 is part of the primary school. Most teachers in the ISP opt to teach in primary schools.

[4] In South African schools learners have a choice between mathematics and mathematics literacy. The learners make the choice between mathematics and mathematics literacy at the end of grade 9 and must complete either one of the two subjects to obtain the grade 12 matriculation certificate.

[5] Ex-model C schools are schools in geographic areas designated for the white population during the apartheid era in South Africa.

[6] Critical inquiry features explicitly in the post-apartheid school curricula.

[7] The following algebraic problem from Dreyfus's(2004) research was given to the students; solve for *n* in $1-\frac{1}{n+2}-(1-\frac{1}{n+2})=\frac{1}{132}$

feedback was given to the students the following day and students were challenged to reflect on the feedback and to resubmit their work on the day the feedback was given. The purpose of the feedback was to engage the students in the importance of working with their productions as opportunities for critical inquiry into the mathematics embedded in their solution. This form of feedback draws on the research of Borasie (1994) in which students' mathematical errors are engaged with to extend the mathematical inquiry to present students with a view of knowledge as a dynamic process of inquiry.

The Problem

The problem was selected from a publication on mathematics competitions by the Association of Mathematics Education for South Africa (AMESA). The problem given below was adapted in the sense that students were asked to do the computation without using the standard algorithm but to apply their knowledge of place value and other properties of the real number system:

$$1997 \times 1995 - 1999 \times 1993$$

The students were also asked not to use calculators. The problem was chosen to see whether students would recognise relationships between the numbers in the numerical structure of the problem and apply their knowledge of the properties of number to find the solution.

RESULTS

None of the first year students and fourth year students was able to produce a correct solution to the problem. Many of the fourth year students were resistant to solve the problem as they did not think that it was possible to solve the problem without following the standard algorithmic route.

The table below shows the solutions of students[8] and the reasons that they gave to support their strategy:

Student	Solution Strategy	Reason
A: 1st Year	Because $(1990 \times 1990) - (1990 \times 1990) = 0$ therefore, $1997 \times 1995 - 1999 \times 1993$ $(7\times5) \times(9\times3)$ $=35 - 27$ $=8$	Student A reasoned that since the numerical structures differed only in their unit digits, the calculation claim which was applied to the one structure must work for the other structure
B:	$(2000 \times 2000) - (2000 \times 2000)$	Student B reasoned that a

[8] The students' solutions are representative of the groups in which they worked. Students who provided reasons for their strategies spoke on behalf of their group.

1^{st} Year	$=(-3\times-5)-(-1\times-7)$ $=8$	quick way to add is to round up but that one must take into consideration what was added
C: 1^{st} Year	$1997 - 1995 = 2$ $1999 - 1993 = 6$ $2 + 6 = 8$	Student C reasoned that the quickest way to calculate is to find the difference and then to add
D: 1^{st} Year E: 4^{th} Year	$(1997-1990)\times(1997-1990)-(1997-1990)\times(1997-1990)$ $=(7\times5)\times(9\times3)$ $==35-27$ $=8$	Students D and E reasoned that if one subtracts 1990 throughput, the numerical expression would not change its value and that $(7\times 5)-(9\times 3)$ would be an equivalent expression of $\mathbf{1997 \times 1995 - 1999 \times 1993}$

Table 1: Students Solution Strategies and Reasons

After the students in the first year class presented their solution strategies to the class, a student remarked that since so many of the groups got the same answer through different strategies, their strategies must be correct. At this point students used a calculator to check the solution to the problem and this provided further conviction to the students that their solution strategies were correct.

Students were encouraged to reflect on additional ways of convincing others that their solution strategies were correct and to present their arguments in the following lesson. Only the fourth year students produced an additional argument for how they would convince others that their solution was correct:

> ‘We let 1990 = a, then $1997 \times 1995 - 1999 \times 1993$ can be written as
>
> $(a + 7)\ (a + 5) - (a + 9)\ (a + 3)$
>
> Now we can just take out the a’s
>
> $\cancel{(a} + 7)\ (a \cancel{+} 5) - \cancel{(a} + 9)\ (\cancel{a} + 3)$
>
> $= (7 \times 5) - (9 \times 3)$
>
> = 8’

ANALYSIS AND DISCUSSION OF STUDENTS' SOLUTIONS

The analysis is structured into two sections. The first section is an analysis of the students' solution strategies. The second section analyses the opportunities for generalisation and justification that emerged and present the moves to different levels of generalisation.

Solution Strategies of Students

The solution strategies of the students suggest that students gave up the knowledge that they have of number properties and tend to focus on the surface features of numerical expression and struggle to recognise the deeper structural features of the numerical expressions. Student A's strategy shows that it is based on surface features, namely the units digits of the expressions. Student A's explanation that the solution to $(1990 \times 1990) - (1990 \times 1990) = 0$ because $(0\times 0) - (0 \times 0)$ is a means to find a justification for the student's strategy. The student may know that $(1990 \times 1990) - (1990 \times 1990) = 0$ because the expression can be generalised to $(x - x)$.

Student B's strategy and reason suggests that the student is aware of strategies such as rounding up and applying the principle of conservation but does not represent the strategy as $(2000 - 3) \times (2000 - 5) - (2000 -1) \times (2000 -7)$ that prompts exploration of the deeper structure of the expression. Student C's strategy does not reflect application of a previously learnt strategy and that the student was simply focussed on generating an answer through operating on the numbers in a way that adhered to the instruction of not applying the standard algorithm.

Students D and E's strategy reflects the application of applying the principle of generating equivalent equations to expressions. The students know that if $x - 1990 = y - 1990$, then $x = y$ and uses this information as the basis on which to argue that if 1990 is subtracted from each of the numbers in the given problem $(1997 - 1990) \times (1995 - 1990) - (1999 - 1990) \times (1993 - 1990)$, then the problem can be reduced to $(7 \times 5) - (9 \times 3)$. The fourth year students persisted in applying this incorrect strategy even when the students introduced a letter to present the problem in a more generalised form as $(a+7)(a+5) - (a+9)(a+3)$.

Opportunities for engaging students in generalisation and justification

A common element that emerged in engaging with the students' reasoning about their strategies is that they did not see the need to check their strategies with problem types that had the same structure. Student A's strategy will be used to illustrate the way in which the students were asked to explore, in a systematic way, whether their strategies could be applied to problem structures with different unit digits. For example, in the case of student A the student was first challenged to see if the strategy worked if and only if one of the last digits was changed to a zero before considering other cases in which the last digit is changed. The table below shows the different problems used to check student A's strategy:

1990 × 1994	3968060	1997 × 1991	3976027	1997 × 1995	3984015	1997 × 1995	3984015
1995 × 1990	3970050	1991 × 1995	3972045	1999 × 1993	3984007	1999 × 1992	3982008
(0 × 4) – (5 × 0)	-1990	(7 × 1) – (1 × 5)	3982	(7 × 5) – (9 × 3)	8	(7 × 5) – (9 × 3)	2007

The systematic process of checking whether the strategy could be applied to other cases did convince the students that for their strategy to be correct it had to work for all cases. But this did not automatically prompt the students to consider what it was about the structure of the problem that allowed their specific strategy to work. Students needed to be asked to focus explicitly on the relationship between the numbers in the specific structure that allowed their strategy to work for the specific problem. Student A first generated the same structure as the fourth year student and then applied the distributive property of numbers to provide a valid argument for the strategy as follows:

'I let 1990 = a

Then $(a + 7) (a + 5) - (a + 9) (a + 3) = a^2 + 12a + 36 - a^2 - 12a - 27 = 8$'

Student A proceeded to show that provided a number has a place value structure in which only the digit value differs, and the sum of the unit digits are equal to twelve in the numbers in each term, the calculation could be reduced to the subtraction of the product of the unit digits. The student illustrated with a specific example as follows:

'I let 34590 = a in the expression $34597 \times 34595 - 34594 \times 34598$

And the expression then becomes $(a + 7) (a + 5) - (a + 4) (a + 8)$'

The student points out that $7 + 5 = 12$ and $4 + 8 = 12$.

Student A has a partial generalisation schema in being only able to see that the unit digits must be equal to twelve. Further generalisation would require the student to see $(a + 7) (a+ 5) – (a+9) (a+3)$ and $(a + 7) (a + 5) - (a + 4) (a + 8)$ as special cases of the general structure $(a+d) (a+e) – (a+f) (a+g)$.

The special case $(a + 7) (a+ 5) – (a+9) (a+3)$ when operated on produces the equivalent structure, $a^2 + 12a + 35 - a^2 - 12a - 27$, which does reveal the generality of the structure since the "a" term will always have equal coefficients if $d + e = f + g$ in the general structure $(a+d) (a+e) – (a+f) (a+g)$. This is an instance of generalisation in which multiple cases are not required as in inductive reasoning which student A displayed (i.e. through trying examples) in an attempt to prove the validity of the calculation strategy. Student A did not realise that the equivalent structure that was produced, $a^2 + 12a + 35 - a^2 - 12a - 27$, reveals that the strategy will only work if and only if digit units differ in the place value structure of the number and that there is a special relationship between the digit units.

The solution to $1997 \times 1995 - 1999 \times 1993$ within the AMESA competition publication provided a different solution strategy which none of the students in the first and fourth year produced:

Let $t = 1997$, then $1997 \times 1995 - 1999 \times 1993$

$$= t(t - 2) - (t + 2) (t - 4)$$

$$= t^2 - 2t - t^2 - 2t + 4t + 8$$

$$= 8$$

The problem posed to the students as part of establishing a classroom inquiry culture was whether they could generate questions that could lead to further exploration of AMESA solution strategy. Most students acknowledged that the school classroom culture is generally one in which they are always given questions for which they have to find solutions but are not expected to generate questions to explore mathematical features of the problem. Students struggled to come up with questions for further exploration. The question, what kind of expressions with the same structure would always generate a constant when using the AMESA strategy, $t (t + a) - (t + b) (t + c)$, was presented to the students. This led to the testing of the AMESA strategy with a specific case in which one of the unit digits was changed (the unit digit of the last number was changed from 3 to 4):

Let $t = 1997$, then $1997 \times 1995 - 1999 \times 1994$

$$= t(t - 2) - (t + 2) (t - 3)$$

$$= t^2 - 2t - t^2 - 2t + 3t + 6$$

$$= -t + 6$$

The testing with a single case did not reveal the relationship between a, b and c in the structure $t (t + a) - (t + b) (t + c)$ that would generate a constant solution for the students as they focussed on the final equivalent expression $-t + 6$. The relationship between a , b and c that will produce a constant can be deduced from the specific case if the students focussed on the following equivalent form $- 2t - 2t + 3t + 6$ and generated the following equivalent form

$t(-2 \; -2 + 3) + 6$ from which they could have argued that $-2 \; -2 + 3$ is not equal to zero and to produce a constant the sum of the three numbers must be zero. This once again illustrates that the generalisation of the strategy for producing the constant did not necessarily require a logically deductive route from the general structure:

$$t(t + a) - (t + b)(t + c)$$

$$= t^2 + at - t^2 - bt - ct - bc$$

$$= at - bt - ct - bc$$

$$= t(a - b - c) - bc$$

Therefore $(a - b - c) = 0$

$\rightarrow \qquad a = b + c$

On establishing the relationship between a, b and c in the structure, $t(t + a) - (t + b)(t + c)$, the students could revisit the special case in which only the last digit of the original problem was changed from 3 to 4 ($1997 \times 1995 - 1999 \times 1994$) and generate different examples changing the last digit as well as other digits that would generate a constant, for example:

Let $t = 1997$, then $1997 \times 1996 - 1999 \times 1994$

$$= t(t - 1) - (t + 2)(t - 3) \qquad \text{...........}[-1 = 2 + -3]$$

$$= t^2 - t - t^2 - 2t + 3t + 6$$

$$= 6$$

CONCLUSION

The incorrect solutions strategies of the students in this case study reflect findings about students' difficulty in applying the properties of numbers to create equivalent forms of number expressions. The key difficulty is the recognition of the relationship between the numbers in the expressions and to use a variable to express the general structure of the number expression. In this case study when students were not able to identify the relationships between numbers in the expression they often apply the properties of number incorrectly simply to get an answer.

CAPS place a strong focus on recognising patterns in number sequences to introduce learners to early experiences in the process of generalisation. In this case study the focus was on engaging pre-service teachers in the process of generalisation in a different kind of number context that involves a numerical expression and recognising its structure to support computation strategies. The challenge for pre-service teachers being prepared for primary school is to consider different kinds of number contexts which support learners in engaging with generalisation processes and to recognise that generalisation and recognising mathematical structure are interrelated.

The incorrect solution strategies of the pre-service teacher students presented a context for challenging them through a critical inquiry approach to reflect on their strategies. In a pre-service teacher training course students' engagement with their mathematics errors is an important component of preparing students to work with learners' errors. Engaging with students' errors in a way that allowed students not only to correct their errors but to consider the role of errors in exploring further mathematics in the problem is important if we want them to extend their learners' thinking through engagement with learners' errors.

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