

A review of Newtonian and non-Newtonian flow in rectangular open channels

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Flumes are ubiquitous throughout the mining and minerals processing industries as well as being used in the polymer processing and pulp and paper industries. With the increased demand for the transport of highly concentrated slurries in rectangular open channels, there is an impetus for a better understanding of the role of rheology in flow hydraulics. Until recently, design of such channels has been based on either the Chézy or Manning equations used for modelling open channel flow of Newtonian fluids. This has been done for modelling turbulent flow of low concentrated slurries with some success. However, this is not the case for highly concentrated slurries where the flow in the channel is no longer turbulent.

Unlike Newtonian fluids, work carried out on flow of pseudo-homogeneous, non-Newtonian fluids in open channels is somewhat limited. Moreover, these results have been published in a variety of sources. This paper attempts to give a critical review of the work carried out on Newtonian and non-Newtonian flow of pseudo-homogeneous non-settling slurries in rectangular open channels. A comparison is also made of the models based on power law, Bingham plastic and Herschel-Bulkley fluids for laminar and turbulent flow in rectangular open channels.

Notation

a	geometric parameter for channel shape	-
A_0	function used by Naik ³¹	-
A_r	area ratio	-
b	geometric parameter for channel shape	-
C	intercept	-
$C_{Chézy}$	Chézy coefficient	$m^{1/2}s^{-1}$
d_{85}	particle size of the solids passing at the 85th percentile	m
D	internal pipe diameter	m
D_h	hydraulic diameter	m
e	roughness height	m
f	Fanning friction factor	-
f_L	Fanning friction factor for laminar flow	-
f_T	Fanning friction factor for turbulent flow	-
Fr	Froude number	-
g	gravitational acceleration	m/s^2
h	flow depth in channel	m
He	Hedström number	-
k	constant	-
k_s	Keulegan constant	m
$k_{Strickler}$	Stricker constant	$m^{1/3}/s$
K	consistency coefficient	$Pa\ s^n$
m	exponent	-
m_{Bazin}	Bazin constant	$m^{1/2}$
M	slope	-
n	power law index	-
n^*	power law index as defined by Kozicki and Tiu ^{21,22}	-
n_{Kutter}	Kutter roughness coefficient	-
$n_{Manning}$	Manning constant	$s/m^{1/3}$
R	radius	m
R_h	hydraulic radius	m
Re	Reynolds number	-
$Re_{2(BP)}$	Reynolds number used by Haldenwang <i>et al.</i> ²⁵ for a Bingham fluid	-
$Re_{2(YPP)}$	Reynolds number used by Haldenwang <i>et al.</i> ²⁵ for a yield pseudoplastic fluid	-
Re_B	Bingham Reynolds number	-
$(Re_B)_c$	critical Bingham Reynolds number	-
Re_c	critical Reynolds number	-
Re_P	power law Reynolds number	-
Re_r	Slatter roughness Reynolds number	-
Re_{Zhang}	Zhang Reynolds number	-
Re^*	Yang and Zhao roughness Reynolds number	-
Re^*_B	Bingham Reynolds number defined by Kozicki and Tiu ^{21,22}	-
Re^*_p	power law Reynolds number defined by Kozicki and Tiu ^{21,22}	-
s	slope	%
V	average velocity	m/s
V_N	Newtonian velocity	m/s
V_{turb}	average velocity for smooth turbulent flow	m/s
V^*	shear velocity	m/s
W	channel width	m
y	exponent	-
α	shape factor defined by Coussot ²⁶	-
ε	Powell's channel roughness factor	-

Notation (continued)

η	viscosity	Pa s
η_{100}, η_{500}	point viscosity at a shear rate of 100 s^{-1} , 500 s^{-1}	Pa s
η_e	equivalent viscosity	Pa s
η_B	Bingham viscosity	Pa s
η_N	Newtonian viscosity	Pa s
θ	angle of inclination to the horizontal	degrees
ρ	fluid or slurry density	kg m^{-3}
τ_w	wall shear stress	Pa
	contour-integrated average values of $\bar{\tau}_w$	Pa
τ_{yB}	Bingham yield stress	Pa
τ_{yHB}	Herschel-Bulkley yield stress	Pa
ϕ_c	ratio of Bingham yield stress to critical wall shear stress	-
χ	$\chi = \tau_{yB} / \bar{\tau}_w$	
Ω	parameter to take into account the blunting of the velocity profile caused by yield stress	-

Introduction

Rectangular channels are used widely in the mining and minerals processing industry for the transport of mineral ore slurries or tailings over long distances and mountainous terrain. Examples of such flumes are those found in Chile and Peru^{1–3}. With water becoming a scarce and expensive resource, there has been a demand for significant reduction in the amount of water used in the preparation of slurries. This, in turn, causes the slurry to become increasingly non-Newtonian in character, thus rendering the models for channel design based on Newtonian flow inappropriate.

As for pipeflow flow, the choice of the design method used for predicting channel flow depends on whether the slurry is *settling* or *non-settling*. For instance, coarse-particle, granular and flocculated slurries tend to settle whereas stabilized fine-particle slurries have a low settling tendency. However, the distinction between non-settling and settling slurries is not that clear-cut as judgement has to be made about the period over which settling is deemed to be important. One possible approach is to assume a given degree of settling that is acceptable before the apparently non-settling slurry must be treated as settling. Non-settling slurries are said to be pseudo-homogeneous if the solid particles are uniformly distributed throughout the carrier fluid⁴. This paper will deal with pseudo-homogeneous, non-settling slurries in rectangular open channels.

Newtonian flow in rectangular open channels

Since the flow of Newtonian fluids in open channels has been well covered in standard textbooks^{5–7}, a brief summary of the various models for laminar and turbulent flow will be given here.

Laminar flow

Laminar flow of Newtonian slurries in open channels is rarely encountered. As for pipe flow, the Fanning friction factor, f , is related to the Reynolds Number, Re :

$$f = \frac{k}{Re} \quad [1]$$

but for open channel flow,

$$f = \frac{2R_h g \sin \theta}{V^2} \quad [2]$$

and

$$Re = \frac{\rho V 4R_h}{\eta} \quad [3]$$

The constant k to be used in Equation [1] is dependent on the channel shape⁵. For open rectangular channels, $k = 16$.

Turbulent flow

The various expressions that have been used for predicting turbulent flow of Newtonian slurries in open channels are listed in Table I. The earliest expression was that introduced by Chézy in 1769 for turbulent flow of water in open channels. Chézy presented several values for the Chézy coefficient, $C_{Chézy}$. Several researchers have assumed that this constant was independent of flow conditions, but research has shown that this is not the case⁷. There are a number of empirical expressions available for the calculation of $C_{Chézy}$. These are also given in Table I.

The Manning equation was derived from the Chézy equation⁹ and is valid for both uniform and non-uniform (gradually varied) flow of water. Dooge¹⁰ pointed out that this expression is improperly called the Manning equation since it was first proposed by Gauckler¹¹ from his reanalysis of the data obtained by Darcy and Bazin¹². Because of this, he suggested the Manning equation should be renamed the Gauckler-Manning equation. This empirical equation was found to be ‘reasonably reliable’ when predicting fully rough turbulent flow of water in open channels⁷. In Europe, the Strickler equation is used in preference to the Manning equation⁷. The exponent of the hydraulic radius in the Manning equation for water flow is actually a variable dependent on the channel shape and roughness⁵. One equation which includes a variable exponent is that proposed by Pavlovskiĭ¹³.

As there has been no or little research on the applicability of these equations to Newtonian fluids other than water, use has also been made of the pipe flow models as a means of predicting turbulent flow of Newtonian slurries in open channels. This is done by replacing the pipe diameter, D in the pipe flow model with the channel hydraulic diameter, D_h which is four times the hydraulic radius, R_h . Pipe flow models that have been adapted using this approach include the well-known Blasius¹⁴, von Karmán-Prandtl^{15,16} and Colebrook-White¹⁷ expressions.

Laminar-turbulent transition

As for pipe flow, the laminar-turbulent transition does not occur at one point but over a range of Reynolds numbers. Moreover, measurements of the critical Reynolds number Re_c conducted by Straub *et al.*¹⁸ for Newtonian flow in rectangular channels varied from 2 000 to more than 3 000 for h/W ratios from 1.35 to 3.70. They also concluded that the values of Re_c for open channel flow, which depend to a certain extent on the channel shape, are generally larger than those for closed conduit flow.

Table I
Various forms of Chézy/Manning equations used for turbulent flow of Newtonian fluids in open channels

Author	Year	Average velocity	Author	Year	$C_{Chézy} \text{ (m}^{1/2}\text{ s}^{-1}\text{)}$	Comment
Chézy	1769	$V = C_{Chézy} \sqrt{R_h \sin \theta}$	Chézy	1769	$C_{Chézy} = 30 \text{ m}^{1/2} \text{ s}^{-1}$ (small rough channel) to $90 \text{ m}^{1/2} \text{ s}^{-1}$ (large smooth channel)	
			Bazin	1865	$C_{Chézy} = \frac{87}{1 + \frac{m_{Bazin}}{\sqrt{R_h}}}$	Values of m_{Bazin} in $\text{m}^{1/2}$ for various types of channels can be found in Darcy and Bazin ⁽¹²⁾
Manning or Gauckler-Manning	1890 1867	$V = \frac{1}{n_{Manning}} (R_h)^{2/3} \sqrt{\sin \theta}$	Ganguillet and Kutter	1869	$C_{Chézy} = \frac{0.00281}{s} + \frac{1.811}{n_{Kutter} \sqrt{R_h}}$ $C_{Chézy} = \frac{1}{1 + \left(41.65 + \frac{0.00281}{s} \right) \frac{n_{Kutter}}{\sqrt{R_h}}}$	Note R_h is in ft and $C_{Chézy}$ is in $\text{ft}^{1/2} \text{ s}^{-1}$. To obtain $C_{Chézy}$ in $\text{m}^{1/2} \text{ s}^{-1}$, multiply by 0.552. Values of n_{Kutter} for various types of channels can be found in Chow ⁽⁵⁾
			Keulegan	1938	$C_{Chézy} = 18.0 \log_{10} \left(\frac{4 R_h}{k_s} \right) + 8.73$	Values of k_s in mm for various types of channels can be found in Keulegan ⁽⁸⁾
			Powell	1950	$C_{Chézy} = -42 \log_{10} \left(\frac{C_{Chézy}}{4 Re} + \frac{\varepsilon}{R_h} \right)$	Values of ε for various types of channels can be found in Chow ⁽⁵⁾
Strickler		$V = k_{Strickler} (R_h)^{2/3} \sqrt{\sin \theta}$			$C_{Chézy} = \frac{1}{n_{Manning}} (R_h)^{1/6}$	Values of $n_{Manning}$ in $\text{s/m}^{1/3}$ used for various types of channels can be found in Chow ⁽⁵⁾
					$C_{Chézy} = k_{Strickler} (R_h)^{1/6}$	Note $k_{Strickler} = 1/n_{Manning}$ ⁽⁷⁾
Chézy	1769	$V = C_{Chézy} \sqrt{R_h \sin \theta}$	Pavlovskii	1937	$C_{Chézy} = \frac{1}{n_{Manning}} (R_h)^y$	$y = 2.5 \sqrt{n_{Manning}} - 0.13$ $- 0.75 \sqrt{R_h} \left(\sqrt{n_{Manning}} - 0.1 \right)$ $0.1 \text{ m} < R_h < 3 \text{ m}, 0.011 < n_{Manning} < 0.04$
			Chow	1959		$y = 1.5 \sqrt{n_{Manning}}$ for $R_h < 1.0 \text{ m}$ $y = 1.3 \sqrt{n_{Manning}}$ for $R_h > 1.0 \text{ m}$

Non-Newtonian flow in rectangular open channels

In contrast to flow of Newtonian fluids in open channels, research with pseudo-homogeneous, non-Newtonian fluids has been somewhat sporadic. Until recently, there has been little interest in developing engineering design models for open channels for fluids exhibiting non-Newtonian behaviour. With the need to concentrate slurries flowing down flumes in the minerals industry as a means of conserving water, a better understanding of the impact of changing the slurry rheology on channel flow hydraulics is now being sought. For some of these slurries, this has meant a change from turbulent flow to laminar flow in the open channel^{19,20}. This paper attempts to give a critical review of the various models available for laminar and turbulent flow of pseudo-homogeneous, non-Newtonian fluids in rectangular open channels.

Laminar flow

The available models in the literature tend to be empirically based owing to the problem of defining the shear stress distribution in an open channel. Table II gives a list of these models together with their applicability to power law (PL), Bingham plastic (BP) or Herschel-Bulkley (HB) models used to describe the slurry flow behaviour. Since these models are for laminar flow only, a check will need to be made beforehand to ensure that the Reynolds number appropriate to the flow model does indeed correspond to laminar flow.

In the late 1960s, Kozicki and Tiu^{21,22} proposed a method based on a generalization of the Rabinowitsch-Mooney equation for pipe flow for calculating the average velocity of uniform flow of various non-Newtonian fluids in open channels of arbitrary cross-section. Expressions for rectangular open channels based on the power law and the Bingham plastic models are given in Table II. The a and b values in these expressions are usually the analytical solutions of Straub *et al.* for Newtonian flow¹⁸. No experimental work was carried out by Kozicki and Tiu^{21,22} to validate their model.

Alternative expressions based on the Bingham plastic flow model have been put forward by Zhang and Ren²³ and Abulnaga²⁴. Zhang and Ren²³ studied the flow of river mud in a 43 m, 20 × 30 cm rectangular concrete open channel with a fixed gradient of 3.66%. In their analysis, they defined a new Reynolds Number, Re_{Zhang} which is similar to the Re_2 definition for a Bingham plastic proposed by Haldenwang *et al.*²⁵. Unfortunately, no indication was given by Zhang and Ren²³ about how well their approach predicted their experimental results. Abulnaga²⁴ modified the Buckingham equation for pipe flow by expressing both Reynolds and Hedström numbers in terms of the channel hydraulic radius. With this approach, he successfully designed a tailings launder in 1997 for a Peruvian copper mine to transport tailings containing soft clay. Paterson *et al.*³ also used his method to analyse the capacity of existing canals from two Andean mines in southern Peru. They reported that the method provided a good approximation of the flow behaviour of the copper tailings in the canals.

For flow of kaolin suspensions in rectangular (and trapezoidal) open channels, Coussot²⁶ put forward a model based on the Herschel-Bulkley model with the power law exponent fixed at $1/3$. He claims that his model is valid for $h/W < 1$. However, this was shown not to be the case for a 300 mm wide rectangular open channel since the shape factor used in the model, α actually reaches a minimum value at a depth of 60 mm, indicating a h/W ratio of about 0.2. Despite this claim, he found his model predicted his rectangular channel experimental wall shear stress data to within 30%.

In analysing their experimental data for CMC solutions and kaolin and bentonite suspensions in rectangular (semi-circular and trapezoidal) flumes, Haldenwang *et al.*²⁵ and Haldenwang²⁷ made use of the Slatter and Lazarus²⁸ pipe Reynolds number with the

Table II
Summary of non-Newtonian, laminar flow models for rectangular open channels

Author	Year	PL	BP	HB	Average velocity	Reynolds number
Kozicki and Tiu ^{21,22}	1967	•			$V = \frac{R_h}{2} \left(\frac{\bar{\tau}_w}{K} \right)^{\frac{1}{n}} \left(\frac{n}{a + bn} \right)$	$Re^*_{*p} = \frac{R_h^n V^{2-n} \rho}{2^{n-3} K \left(\frac{a + bn}{n} \right)^n}$
					$V = \frac{R_h}{2} \frac{\bar{\tau}_w}{\eta_B} \left[\frac{1}{a+b} - \frac{\chi}{b} + \frac{a}{b(a+b)} \chi^{\frac{b}{a+1}} \right]$	$Re^*_{*B} = \frac{4R_h V}{\eta_B} \left[\frac{1}{a+b} - \frac{\chi}{b} + \frac{a}{b(a+b)} \chi^{\frac{b}{a+1}} \right]$
Zhang and Ren ²³	1982		•		$V = \frac{R_h}{2} \frac{\tau_w}{\eta_B} \left[1 - \frac{\tau_{yB}}{\tau_w} \right]$	$Re_{Zhang} = \frac{4R_h \rho V}{\eta_B + \frac{\tau_{yB} R_h}{2V}}$
Abulnaga ²⁴	2002		•		$V = \frac{R_h}{2} \frac{\tau_w}{\eta_B} \left[1 - \frac{4\tau_{yB}}{3\tau_w} \right]$	$Re_B = \frac{4R_h V \rho}{\eta_B}$
Coussot ²⁶	1994			•	$V = \frac{h}{\left[\frac{K}{\tau_{yHB}} \left\{ \left(\frac{a}{(\tau_w/\tau_{yHB}) - 1} \right)^{1.11} \right\} \right]^{\frac{1}{n}}} \text{ with } n = \frac{1}{3}$ $\alpha = 1.93 - 0.43 \left(\arctan \left(\left(\frac{10h}{W} \right)^{20} \right) \right) \text{ for } \frac{h}{W} < 1$	
Haldenwang ^{25,27}	2002	•			$V = \frac{R_h}{2} \left[\frac{\tau_w}{K} - \frac{\tau_{yHB}}{K} \right]^{\frac{1}{n}}$	$Re_{2(YPP)} = \frac{8\rho V^2}{\tau_{yHB} + K \left(\frac{2V}{R_h} \right)^n}$

$4R_h$ in place of the pipe diameter. They found that their approach adequately predicts their experimental data in the laminar flow regime, provided the correct equation is used to take account of the rheology of the test fluid used.

Turbulent flow

The various models available in the literature for predicting the friction factor of non-Newtonian fluids in turbulent open channel flow are presented. A summary of these models is given in Table III for smooth and rough walled open channels. With the exception of the Manning equation, these models have been developed for this type of flow from the pipe flow models where the pipe diameter was replaced by four times the hydraulic radius.

The Manning equation discussed earlier for Newtonian open channel flow is included in Table III since it is still being used in the minerals industry for designing flumes transporting non-Newtonian slurries in turbulent flow. For example, Fuentes *et al.*¹ report the use of this approach to design flumes for the transportation of tailings in South America. This approach is thought to be valid when the fluid is described as being near-Newtonian. However, this becomes questionable when the fluid becomes more non-Newtonian in character.

Table III
Summary of non-Newtonian, turbulent flow models for smooth and rough-walled, rectangular open channels

Author	Year	PL	BP	HB	Friction factor	Average velocity		Reynolds number	
Smooth-walled rectangular open channels									
Manning or Gauckler-Manning ^(9,11)	1890 1867				$f = \frac{2R_h g \sin \theta}{V^2}$	$V = \frac{1}{n_{Manning}} (R_h)^{\frac{1}{3}} \sqrt{\sin \theta}$			
Kozicki and Tiu ^(21,22)	1967	•			$\frac{1}{\sqrt{f}} = \frac{4.0}{n^{0.75}} \log \left[\text{Re}_p^* f^{\left(\frac{1-n}{2}\right)} \right] - \frac{0.40}{n^{1.2}} + 4.0 n^{0.25} \log \left[\frac{4(a+bn)}{(1+3n)} \right]$		$Re_p^* = \frac{R_h^n V^{2-n} \rho}{2^{n-3} K \left(\frac{a+bn}{n} \right)^n}$		
Abulnaga ⁽²⁴⁾	2002		•		$f = (f_L^m + f_T^m)^{(1/m)}$ $f_L = \frac{16}{\text{Re}_B} \left[1 + \frac{He}{6\text{Re}_B} + \frac{He^4}{3f_L \text{Re}_B^2} \right]$ $f_T = 10^a \text{Re}_B^b$ $a = -1.47 \left(1 + 0.146 \exp \left(-2.9 \times 10^{-5} \left(\frac{16R_h^2 \rho \tau_w}{\eta_B^2} \right) \right) \right)$ $b = -0.193$ $m = 1.7 + \frac{10000\eta_B}{R_h V \rho}$		$Re_B = \frac{4R_h V \rho}{\eta_B}$		
Wilson and Thomas ^(32,33)	1985	•	•	•	$f = \frac{2R_h g \sin \theta}{V^2}$	$V = V_* \left(\frac{V_N}{V_*} + 11.6(A_r - 1) - 2.5 \ln A_r - \Omega \right)$ $A_r = 2 \frac{1 + (\tau_y / \tau_w)n}{1 + n}$ $V_N = V_* \left(2.5 \ln \left(\frac{\rho V_* 2R_h}{\eta_e} \right) + 1.75 \right)$ $\Omega = -2.5 \ln \left(1 - \frac{\tau_y}{\tau_w} \right) - 2.5 \frac{\tau_y}{\tau_w} \left(1 + 0.5 \frac{\tau_y}{\tau_w} \right)$	$V_* = \sqrt{\frac{\tau_w}{\rho}}$	$Re = \frac{4R_h V \rho}{\eta_e}$	
Torrance ⁽³⁴⁾	1963	•	•	•	$f = \frac{2R_h g \sin \theta}{V^2}$	$\frac{V}{V_*} = \frac{3.8}{n} + \frac{2.78}{n} \ln \left(1 - \frac{\tau_y}{\tau_w} \right) + \frac{2.78}{n} \ln \left(\frac{V_*^{2-n} \rho (2R_h)^n}{K} \right) - 4.17$		$Re = \frac{8\rho V^2}{K \left(\frac{2V}{R_h} \right)^n}$	

Table III
Summary of non-Newtonian, turbulent flow models for smooth and rough-walled, rectangular open channels (continued)

Author	Year	PL	BP	HB	Friction factor	Average velocity	Reynolds number
Slatter ⁽³⁵⁾	1994	•	•	•	$f = \frac{2R_h g \sin \theta}{V^2}$	$V = V_* \left(2.5 \ln \left(\frac{2R_h}{d_{85}} \right) + 2.5 \ln Re_* + 1.75 \right)$ provided $Re_* \leq 3.32$	$Re_* = \frac{8\rho V_*^2}{\tau_y + K \left(\frac{8V_*}{d_{85}} \right)^n}$
Haldenwang ⁽²⁷⁾	2003	•	•	•	$f = \frac{0.66(2gh \sin \theta)}{(V_{turb})^2}$	$V_{turb} = \sqrt{gh \sin \theta} \left(2.5 \ln \frac{2R_h}{e} - 76.86 \eta_{500} - 9.45 \right)$	$Re_{\eta(pp)} = \frac{8\rho V^2}{\tau_{yHB} + K \left(\frac{2V}{R_h} \right)^n}$
Rough-walled rectangular open channels							
Manning or Gauckler-Manning ^(9,11)	1890 1867				$f = \frac{2R_h g \sin \theta}{V^2}$	$V = \frac{1}{n_{Manning}} (R_h)^{\frac{2}{3}} \sqrt{\sin \theta}$	
Naik ⁽³¹⁾	1983		•		$f = \frac{2R_h g \sin \theta}{V^2}$	$V = 2.5V_* \left(1 - \frac{\tau_y}{\tau_w} \right)^{0.5} \left(A_0 + \ln \left(\frac{R_h}{e} \right) \right) V_* = \left(\frac{\tau_w}{\rho} \right)^{0.5}$ $A_0 = \ln \left(\frac{30h}{R_h} \right) \left(-1 + \left(\frac{\left(\frac{W+2}{h} \right)^2}{4Wh} \right) \right)$	$Re_B = \frac{4R_h V \rho}{\eta_B}$
Yang and Zhao ⁽³⁷⁾	1983				$\frac{1}{\sqrt{f}} - 2 \log \frac{R_h}{e} = 13.81 - 4.71 \log Re^*$		$Re^* = \frac{\rho V_* e}{\eta}$
Slatter ⁽³⁵⁾	1994	•	•	•	$f = \frac{2R_h g \sin \theta}{V^2}$	$V = V_* \left(2.5 \ln \left(\frac{2R_h}{d_{85}} \right) + 4.75 \right)$ provided $Re_* > 3.32$	$Re_* = \frac{8\rho(V_*)^2}{\tau_y + K \left(\frac{8V_*}{d_{85}} \right)^n}$

The earliest adaptation of the pipe turbulent flow model for open channel flow was that of Kozicki and Tiu^{21,22}. They proposed that the friction factor for turbulent flow of a power law fluid in smooth-walled open channels of arbitrary cross-section may be evaluated using the Dodge and Metzner²⁹ equation for circular pipes recast in terms of n^* , Re_p^* , and the geometric parameters, a and b .

An expression based on the Bingham plastic flow model was put forward by Abulnaga²⁴ who adapted the Darby and Melson³⁰ expression for turbulent pipe flow. He used this approach to design a tailings launder to transport tailings rich in soft clay for a Peruvian copper mine. By adapting the criteria for turbulent flow of a Newtonian fluid in a rough wall open channel of semi-circular cross-section, Naik³¹ derived the mean velocity of a Bingham plastic fluid flowing in a rough open channel of a rectangular cross-section. Using kaolin slurries in a 12.2 m steel flume of a 300 × 300 mm rectangular cross-section tilted at various slopes up to a maximum slope of 1 in 20, he found good reasonable agreement between the experimental data and his model.

An approach based on the Herschel-Bulkley flow model was suggested by Wilson⁴. He proposed the use of the Wilson and Thomas^{32,33} model provided that the pipe radius is replaced by the equivalent hydraulic radius of the open channel and the equivalent viscosity, η_e is used in place of the Newtonian viscosity, η_N . This model is based on the increase in the thickness of the laminar sub-layer caused by the much lower turbulent friction factors found for pipeflow of non-Newtonian fluids than for Newtonian fluids. Alternative expressions for pipe flow based on the Herschel-Bulkley flow model that have been put forward include the Torrance³⁴ model for smooth pipes and the Slatter³⁵ model for smooth and rough pipes.

Wan and Wang³⁶ refer to the work carried out by Yang and Zhao³⁷ on hyperconcentrated flow over a rough flume surface. Based on their experimental results, Yang and Zhao³⁷ obtained an equation for the friction factor based on their definition of the roughness Reynolds Number, Re^* . Yang and Zhao³⁷ also proposed that for smooth boundary hyperconcentrated flow in open channels, the Blasius equation could be used. It appears that the materials tested all have a low yield stress.

Haldenwang²⁷ presented a model based on Slatter's model³⁵ for turbulent flow of a Herschel-Bulkley fluid in a smooth walled pipe. His expressions for the average velocity and friction factor made use of a point viscosity at a shear rate of 500 s⁻¹. This model predicted the velocity for his dataset for kaolin suspensions in a 10 m by 150 mm wide rectangular tilting flume²⁰ and the Naik³¹ dataset for clay suspensions in a 10 m by 300 mm wide rectangular flume with a maximum slope of 1:20 to within 30%.

Laminar-turbulent transition

The various models available in the literature for predicting the laminar-turbulent transition of non-Newtonian fluids in open channels are summarized in Table IV. As for turbulent flow, most of these models have been derived from pipe flow models.

There has been no work reported on transitional open channel flow for fluids exhibiting power law behaviour. Reliance will have to be made of Haldenwang's transition model^{27,38}, which is independent of the fluid rheology, to determine the critical Reynolds number at the onset of transition from laminar to turbulent flow.

Three alternative expressions are available for predicting the critical Reynolds number for the onset of transition flow of Bingham plastic fluids in open rectangular channels. The first is an adaptation of Hanks^{39,40} criterion for the onset of laminar-turbulent transition for

Table IV
Summary of various critical Reynolds numbers used for the determination of the laminar-turbulent transition in rectangular open channels

Author	Year	PL	BP	HB	Critical Reynolds number
Hanks ^(39,40)	1963		•		$(Re_B)_c = \frac{\rho V 4R_h}{\eta_B} = \frac{1 - \frac{4}{3}\phi_c + \frac{\phi_c^3}{3}}{8\phi_c} He$ $\frac{\phi_c}{(1-\phi_c)^3} = \frac{He}{16800} \quad He = \frac{16R_h^2 \rho \tau_{yB}}{\eta_B^2}$
Hao <i>et al</i> ⁽⁴¹⁾	1980		•		$Re_c = \frac{\rho V 4R_h}{\eta + \left(\frac{\tau_y R_h}{2V} \right)}$
Naik ⁽³¹⁾	1983		•		$(Re_B)_c = \frac{\rho V 4h}{\eta_B} = \left(\frac{He}{12\phi_c} \right) (1 - 1.5\phi_c + 0.5\phi_c^3)$ $\frac{\phi_c}{(1-\phi_c)^3} = \frac{He}{48000} \quad He = \frac{16R_h^2 \rho \tau_{yB}}{\eta_B^2}$
Haldenwang ^(27,38)	2003	•	•	•	$Re_c = \frac{200}{\eta_{100}^{0.21}} Fr + \frac{71}{\eta_{100}^{0.75}}$ $Fr = \frac{V}{\sqrt{gh}}$

pipe flow of a Bingham plastic fluid where D is taken to be $4R_h$. The second uses a similar approach taken by Naik³¹ to obtain a different expression for the critical Reynolds number. The third is that of Hao *et al.*⁴¹ who carried out research on transitional flow of hyperconcentrated sediment laden water in rectangular (trapezoidal and U-shaped) channels. Based on his critical Reynolds number definition, he found this to be 48000.

Haldenwang^{27,38} developed a transition model based on the Froude number and the point viscosity of the fluid at 100 s⁻¹ from his open channel work with CMC solutions and kaolin and bentonite suspensions. In plotting the Reynolds number, $Re_{2(YPP)}$ against the Froude number, he observed a trend that was similar for all the materials tested and the onset of transition is easily distinguishable for each slope. With the onset of transition for each slope deemed to be the point of inflection of the appropriate Reynolds number versus Froude number curve, he found that this point corresponded in all cases with the deviation from the $16/Re$ line on the Moody chart. By fitting a curve through the points of inflection for each slope, a transition locus was then obtained for the material under test. This curve was found to be a straight line of the form $Re_c = M Fr + C$ where Re_c is the critical Reynolds number, Fr is the Froude number at the onset of transition, M is the slope and C is the intercept. For the materials tested, both M and C were found to be related to the apparent viscosity at 100 s⁻¹.

Model comparison

Laminar flow

The various models available for the prediction of laminar flow of power law, Bingham plastic and Herschel-Bulkley fluids in rectangular, smooth-wall, open channels are compared using the data from the database recently published by Haldenwang and Slatter⁴². This database consists of the experimental data obtained by Haldenwang²⁷ for the flow of carboxymethylcellulose (CMC) solutions, kaolin suspensions and bentonite suspensions at various concentrations in three rectangular tilting flumes ranging from 75 to 300 mm in width. Rheological characterization of these three fluids was carried out using an in-line tube viscometer with three different diameter tubes⁴².

Figure 1 gives a comparison between the Kozicki and Tiu^{21,22} and Haldenwang²⁵ models for predicting laminar flow of power law fluids in a rectangular channel. The Fanning friction factor was computed from Equation [2] using measured values for R_h and V . The Reynolds number in the plot corresponds to the Re definition used by the model.

Despite the excellent alignment of these two models with the $16/Re$ line, the Haldenwang²⁵ model is the better of the two for predicting the actual velocity of the fluid down the channel as shown in Figure 2. The actual velocity is simply the volumetric flow rate divided by the flow cross-sectional area whereas the model velocity is given by the appropriate equation in Table II for the model using measured values of R_h and τ_w ($= R_h \rho g \sin \theta$).

For laminar flow of Bingham plastic fluids in rectangular channels, the four available models are compared in Figure 3. The Fanning friction factor was calculated from Equation [2] as for Figure 1. The Reynolds number in Figure 3 corresponds to the Re definition used by the model. Only the Zhang and Ren²³ and Haldenwang²⁵ models have perfect alignment with the $16/Re$ line with their data well within the

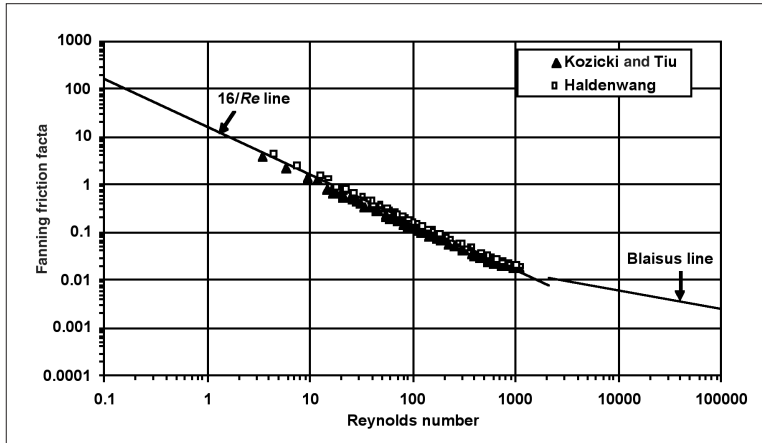


Figure 1. Comparison of power law models for laminar flow of 3.8% CMC solution ($K = 0.606$ Pa sⁿ, $n = 0.68$) in a 300 mm rectangular channel

laminar flow regime. This is in spite of the poor agreement shown in Figure 4 between the model and actual velocities. These velocities were calculated in a similar manner used for Figure 2. A possible cause of the larger scatter observed in Figure 4 is the introduction of the Bingham yield stress as a parameter in the calculation of the model velocities.

The large deviation of the Kozicki and Tiu's model^{21,22} from the $16/Re$ line, despite their data being within the laminar flow regime, could be ascribed to the poor match of model velocities with actual velocities. The failure of the Abulnaga's model²⁴ observed in Figure 3 is due to the assumption that $\tau_{yB}/\tau_w < 0.5$ needed for the simplification of the Buckingham equation for pipe flow not being met. Abulnaga²⁴ makes no mention of this requirement when he presented his expressions for channel flow.

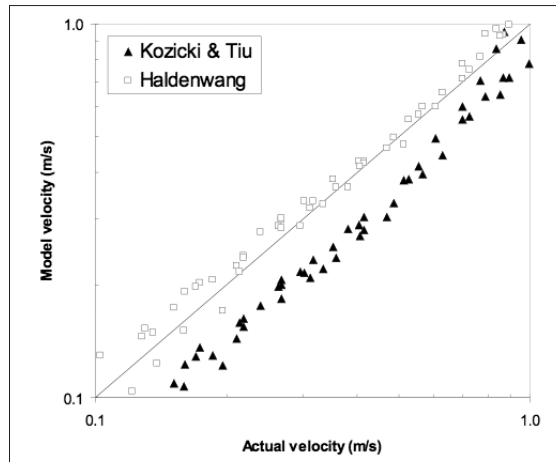


Figure 2. Comparison of model velocities with actual velocities with respect to Figure 1

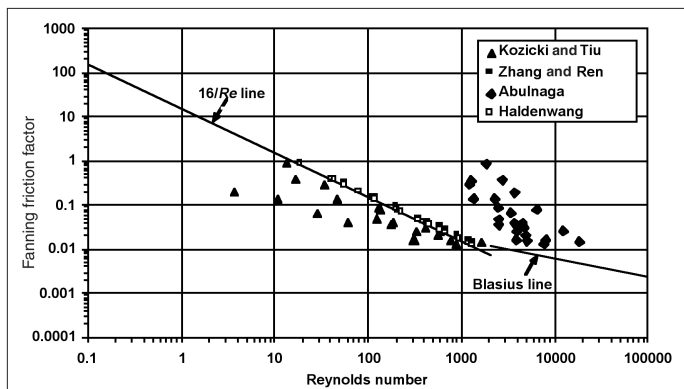


Figure 3. Comparison of Bingham plastic models for laminar flow of 4.5% bentonite suspension ($\tau_y = 4.4$ Pa, $\eta_p = 0.0061$ Pa s) in a 150 mm rectangular channel

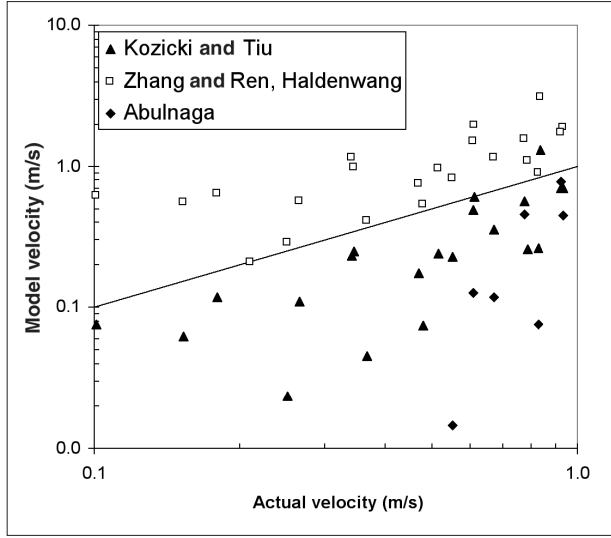


Figure 4. Comparison of model velocities with actual velocities with respect to Figure 3

The two models available for laminar flow of Herschel-Bulkley fluids in rectangular channels are compared in Figure 5. The Fanning friction factor was calculated from Equation [2] as for Figures 1 and 3. It can be seen that both Coussot²⁶ and Haldenwang²⁵ models both lie on the $16/Re$ line. It must be noted that the Re definition used by Haldenwang²⁵ was also used for the Coussot's model²⁶. Despite the scatter shown in Figure 6, Coussot's model appears to be the better model in terms of predicting actual velocities. This scatter is thought to be due to the introduction of the Herschel Bulkley yield stress as a parameter in the calculation of the model velocities.

Turbulent flow

Comparison of the various models for predicting turbulent flow in rectangular, smooth-wall, open channels using the data from the database recently published by Haldenwang and Slatter⁴² will be confined to Herschel-Bulkley fluids only. Details of the various models for the turbulent flow prediction of power law fluids in smooth channels and Bingham plastic fluids in smooth and rough channels have been reported elsewhere^{20,27}.

For fully turbulent flow of Herschel-Bulkley fluids in smooth rectangular channels, Figure 7 gives a comparison of the four available models with the Manning equation with $n_{Manning}$ taken as $0.008 \text{ s/m}^{1/3}$. Any data lying on the $16/Re$ line corresponding to laminar flow and in the transitional flow regime were ignored here. As for laminar flow, the Fanning friction factor was obtained from Equation [2] using measured values for R_h and V whereas the Reynolds number corresponds to the Re definition used by the model. Three different Re definitions were used. The Torrance³⁴ model makes use of the Clapp Reynolds number as defined in Table III, whereas the Wilson-Thomas^{32,33} model uses a Reynolds number based on an equivalent viscosity given by

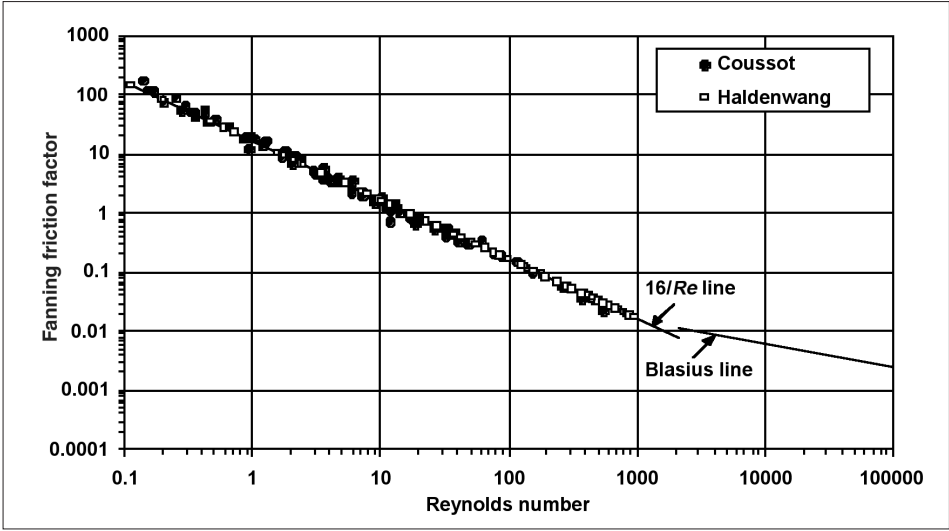


Figure 5. Comparison of Herschel-Bulkley models for laminar flow of 10% kaolin suspension ($\tau_y = 21.3 \text{ Pa}, K = 0.524 \text{ Pa s}^n, n = 0.47$) in a 300 mm rectangular channel

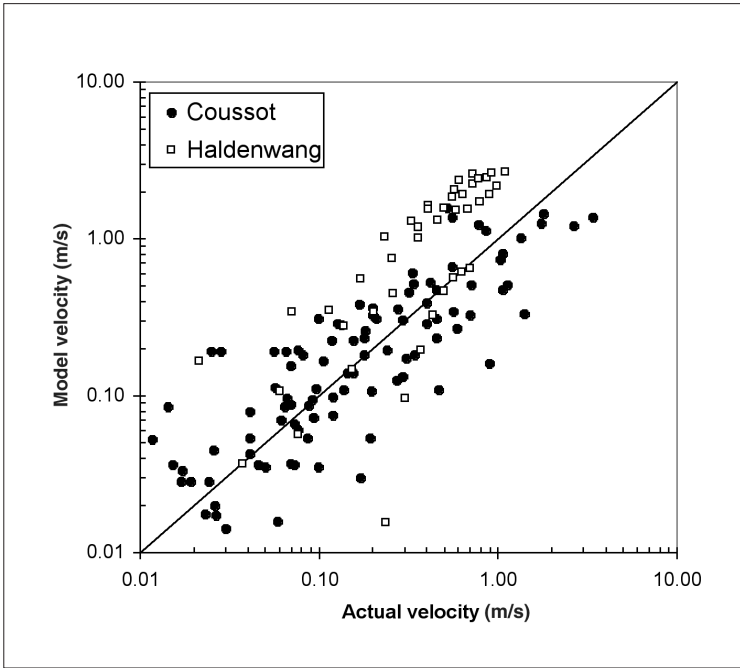


Figure 6. Comparison of model velocities with actual velocities with respect to Figure 5

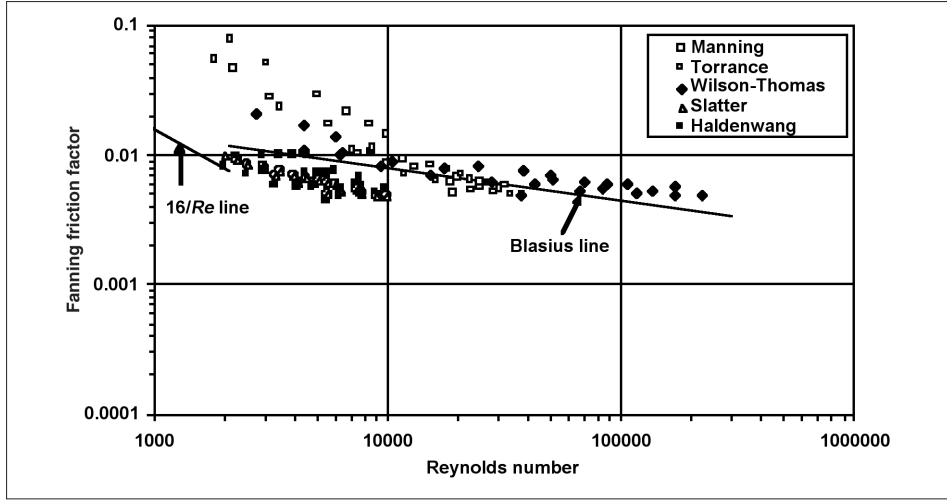


Figure 7. Comparison of Herschel-Bulkley models for turbulent flow of 6% kaolin suspension ($\tau_y = 6.8 \text{ Pa}$, $K = 0.149 \text{ Pa s}^n$, $n = 0.52$) in a 150 mm smooth rectangular channel

$$\eta_e = \frac{\tau_w}{\left(\frac{\tau_w - \tau_y}{K} \right)^{\frac{1}{n}}} \quad [4]$$

The $Re_{2(YPP)}$ definition used by Haldenwang²⁷ for his model was also used in the plot for the Manning⁹ and Slatter³⁵ models.

Apart from the Torrance model, all of the other models give a similar slope to that for the Blasius equation for fully turbulent Newtonian flow given by the solid line in Figure 7. Despite having the widest range of Re of all the models, the Wilson-Thomas model does appear to have the largest amount of scatter compared with the other models. However, a closer scrutiny of the data has revealed that this is actually a slope effect with the upper data points in Figure 7 corresponding to channel slopes of 3, 4 and 5° and the lower data points corresponding to 1 and 2°. This slope effect was much less noticeable in the other models.

Figure 7 also shows the Manning⁹ equation, which was originally developed for turbulent flow of water, to be largely similar to the other four models. This is in spite of the lack of rheological parameters in the equation. This observation does explain why this equation has sometimes been successfully applied for open channel flow of non-Newtonian slurries in the minerals industry¹.

It can be seen from Figure 8 that the Manning⁹, Torrance³⁴ and Slatter³⁵ models predict velocities that are higher than the actual velocities used. Despite the scatter, both the Wilson-Thomas^{32,33} and Haldenwang²⁷ models appear to predict the velocities close to the actual velocities used. Of these two models, the Haldenwang²⁷ model appears to have the least scatter.

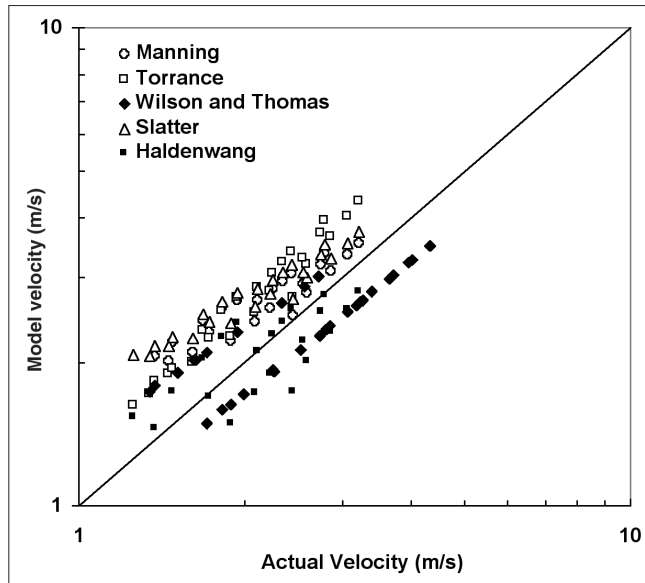


Figure 8. Comparison of model velocities with actual velocities with respect to Figure 7

Conclusions

Work on Newtonian and non-Newtonian flow of pseudo-homogeneous, non-settling slurries in rectangular open channels have been reviewed. The various models available for laminar, transitional and turbulent flow of these slurries have been presented and discussed. Using the channel flow data from the Haldenwang and Slatter⁴² database, the models for laminar flow of power law, Bingham plastic and Herschel-Bulkley fluids in open rectangular channels were compared. This was also done for turbulent flow of Herschel-Bulkley fluids in smooth-wall, open rectangular channels.

It is clear from this limited comparative study that further research is needed to pin down the discrepancies found between the predicted and actual velocities. However, the models developed by Haldenwang *et al.*²⁵ for laminar flow of power law, Bingham plastic and Herschel-Bulkley fluids appear to be the most reliable in terms of predicting actual velocities as well as aligning to the $16/Re$ line. Similarly, the model developed by Haldenwang²⁷ for turbulent flow of Herschel-Bulkley fluids was found to be reliable in predicting actual velocities.

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