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RESEARCH

Linearization of Newton's Second Law

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Abstract
The geometric linearization of nonlinear differential equation is a robust method for the construction of analytic solutions. The method is related to the existence of Lie symmetries which can be used to determine point transformations such that to write the given differential equation in a linear form. In this study we employ another geometric approach and we utilize the Eisenhart lift to geometric linearize the Newtonian system describing the motion of a particle in a line under the application of an autonomous force. Our findings reveal that for the oscillator, the Ermakov potential with or without the oscillator term, and the Morse potential, Newton's second law can be globally expressed in the form of that of a free particle. This study open new directions for the geometric linearization of differential equations via equivalent dynamical systems.

Keywords Newtonian physics · Geometric linearization · Eisenhart lift

1 Introduction

Sophus Lie's theory [1–3] stands as a powerful mathematical tool for the analytic treatment of nonlinear differential equations. A Lie symmetry vector serves as the generator of transformations that leave a given differential equation invariant. Consequently, there exist invariant functions, known as Lie invariants, which are invaluable for simplifying sets of differential equations through reduction [4–7]. The Lie symmetry analysis offers a systematic approach to studying differential equations and has been instrumental in deriving numerous significant results. As a result, it has found widespread use in the analysis of various dynamical systems [8–25].

An intriguing property of Lie symmetry analysis is its ability to globally linearize maximally symmetric differential equations [9, 26]. Second-order ordinary differential equations

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that are invariant under the elements of the $sl(3, R)$ Lie algebra are considered maximally symmetric [11]. Consequently, there always exists a point transformation, or change of variables, where the equation can be globally expressed in the form of a free particle. This linearization process allows for the determination of conservations and the analytic solution for the original equation through a simple application of the point transformation. Due to this significant property, the process of linearization has been extensively explored in the literature, leading to the establishment of various criteria [27–31]. Due to the specific property that a family of polynomial second-order differential equations is related to geodesic equations in [32, 33] some geometric criteria have been established for the linearization, see also the discussion in [34, 35].

In this study we are interested in the linearization properties of the Newtonian system with equation of motion

$$\ddot{x} - F(x) = 0. \tag{1}$$

The latter equation describes the motion for a classical particle under the application of an autonomous force.

The Lie symmetry analysis for (1) has been performed by Sophus Lie, and when $F(x) = F_1 x + F_0$, the differential equation is maximally symmetric, that is, it admits eight Lie symmetries the elements of the $sl(3, R)$; straightforward (1) is a linear equation [13]. For other functional forms of the force $F(x)$ (1) admits less Lie symmetry vectors. Indeed, for $F(x) = (\alpha + \beta x)^n$, $n \neq 0, 1, -3$ and $F(x) = e^{\gamma x}$, $\gamma \neq 0$, the admitted Lie symmetries of (1) are two, and form the algebra A_2 in the Mubarakzyanov Classification Scheme. Moreover, for $F(x) = \alpha(x + c) + \frac{\beta}{(x+c)^3}$, $\beta \neq 0$ (1) admits three Lie symmetries which form the three-dimensional algebra, $A_{3,8}$, which is more commonly known as $sl(2, R)$. Finally, for arbitrary function $F(x)$, (1) admits a unique symmetry, which defines the invariance in time translations.

In the following, we introduce a novel approach where we demonstrate that (1) can be linearized for specific functions of the force, thereby breaking the condition of maximally symmetric equations. To achieve this, we utilize the geometric framework established by L. Eisenhart [40]. We perform a lift in the dimension of (1), transforming it into a system expressed as geodesic equations on a new (pseudo-) Riemannian manifold. This transformation allows the force term $F(x)$ to be attributed to the geometric properties of this manifold.

The Eisenhart lift has found application in a range of studies for the analysis of classical and quantum systems [41–46]. In a related work [47], authors discuss the equivalence of the oscillator with the free particle using the Eisenhart lift. Building upon this foundation, we extend these results by presenting a systematic approach. Specifically, we demonstrate that for specific nonlinear functions $F(x)$, the analytic solution of (1) corresponds to the solution of the free particle. The outline of the paper is as follows.

In Section 2 we present the Eisenhart lift and we define new higher-order lifts. Section 3 includes the main results of this analysis where we employ results from differential geometry in order to determine the functional forms of the force term $F(x)$ where (1) can be linearized via the Eisenhart lift. Finally, in Section 4 we draw our conclusions.

2 Eisenhart Lift

In this section, we delve into the Eisenhart lift, a technique that provides a geometric representation of dynamical systems, allowing them to be formulated as sets of geodesic equations [40]. Through the Eisenhart lift (also known as the Eisenhart-Duval lift), we introduce an equivalent dynamical system of higher-dimension, that is, we introduce new dependent variables. The new dimensions allows us to express the original dynamical system in the equivalent form of geodesic equations for an extended manifold. The extended manifold possesses a sufficient number of isometries, associated with conservation laws, which are essential for recovering the original dynamical system. This procedure is known as the non-relativistic limit of the Kaluza-Klein framework. In [48] this approach was applied to reformulate the equations of motions for particles in the classical and quantum regimes in a covariant form. Hence, via the Eisenhart lift we reveal the solution for the original dynamical system into the solution for the equivalent system of geodesic equations. In the following lines with describe lift in one-dimensional systems of the form of (1).

We define the momentum $p_x = \dot{x}$, thus, (1) is expressed by the equivalent first-order dynamical system

$$\dot{x} = p_x \ , \ \dot{p}_x = F(x) , \tag{2}$$

The latter equations are the Hamilton's equation for the function

$$H = \frac{1}{2}p_x^2 + V(x) \ , \ F(x) = -V(x)_{,x} . \tag{3}$$

Hamiltonian (3) describes the motion of a particle in the one-dimensional space with line element $ds^2 = dx^2$, under the potential $V(x)$. Function (3) is a conservation law for the equations of motion (2) with value $H = h$.

2.1 Equivalent Hamiltonian H_{1+1}

We employ the Eisenhart lift and we introduce the new Hamiltonian function

$$H_{1+1} \equiv \frac{1}{2}p_x^2 + \frac{\alpha}{2}V(x)\, p_z^2 = h_{1+1}. \tag{4}$$

This function describes the geodesic equations for the two-dimensional space with line element

$$ds_{(1+1)}^2 = dx^2 + \frac{1}{\alpha V(x)}dz^2 , \tag{5}$$

which is known as the Eisenhart metric

The geodesic equations for the two-dimensional line element (5) are

$$\dot{x} = p_x \ , \ \dot{z} = \alpha V(x)\, p_z , \tag{6}$$

$$\dot{p}_x = -\frac{\alpha}{2}V_{,x}p_z^2 \ , \ \dot{p}_z = 0. \tag{7}$$

We observe that momentum $p_z = p_z^0$ is a conservation law for the geodesic equations. Hence, by replacing in the rest of the equations we end with the reduced system

$$\dot{x} = p_x \ , \ \dot{p}_x = -\frac{\alpha}{2}V_{,x}\left(p_z^0\right)^2 \ , \ \dot{z} = \alpha V(x)\, p_z . \tag{8}$$

Hence, the original system (2) is recovered when $h_{1+1} = h$ and $\alpha\left(p_z^0\right)^2 = 2$.

However, it is important to note that the extended Hamiltonian in the Eisenhart lift is not uniquely defined. While Hamiltonian (4) represents the most common geometric description for the dynamical system (1), there exist alternative Hamiltonian functions as well.

At this point it is important to remark that the third equation in (8), that is, for the extended variable z, plays no role in the evolution of the original system. This dynamical system is foliated and the set of initial conditions for the variable z, do not affect the evolution of the original system in the variables x, p_x. That property remains valid and for the equivalent Hamiltonians that we discuss bellow.

2.1.1 Equivalent Hamiltonian H_{1+2}

Consider the extended Hamiltonian function

$$H_{1+2} \equiv \frac{1}{2}p_x^2 + V(x)\, p_u^2 + p_u p_v = h_{n+2}, \tag{9}$$

with equations of motion

$$\dot{x} = p_x\ ,\ \dot{u} = 2V(x)\, p_u + p_v\ ,\ \dot{v} = p_u, \tag{10}$$
$$\dot{p}_x = -V_{,x} p_u^2\ ,\ \dot{p}_u = 0\ ,\ \dot{p}_v = 0. \tag{11}$$

Momentum p_u and p_v are conservation laws. By replacing in the rest of the equations we end with the Hamiltonian system (3), assuming that

$$p_u^2 = 1 \text{ and } p_u p_v - h_{1+2} = h. \tag{12}$$

The corresponding Eisenhart metric is described by the pp-wave spacetime

$$ds^2_{(1+2)} = dx^2 + 2dudv - 2V(x)\, du^2. \tag{13}$$

If we select $h_{1+2} = 0$, then (10), (11) correspond to the null geodesics of space (13). As we shall discuss in the following Section null geodesics have the property to be invariant under conformal transformations. This characteristic has been applied before to discover the original of the high-dimensional symmetry group in dynamical systems [49]. For possible extensions we refer the reader to the recent discussion in [50].

The two lifts discussed thus far are commonly applied in the literature. The first one is known as the Riemannian lift [44], while the second one is referred to as the Lorentzian lift [44].

2.2 New Extended Hamiltonians

We proceed with the introduction of two new lifts which, as we shall see, can be applied to linearize (1) for specific functions of $F(x)$.

2.2.1 Equivalent Hamiltonian H_{1+3}

Assuming the extended Hamiltonian function

$$H_{1+3} \equiv \frac{1}{2}p_x^2 + \frac{\alpha}{2}F_1(x)\, p_z^2 + F_2(x)\, p_u^2 + p_u p_v = h_{n+3}, \tag{14}$$

which is a combination of the Riemannian and Lorentzian lifts. The equations of motion are given by

$$\dot{x} = p_x\ ,\ \dot{z} = \alpha F_1(x)\, p_z\ ,\ \dot{u} = 2F_2(x)\, p_u + p_v\ ,\ \dot{v} = p_u\ , \tag{15}$$
$$\dot{p}_x = -\left(\frac{\alpha}{2}F_1(x)_{,x}\, p_z^2 + F_2(x)_{,x}\, p_u^2\right)\ , \tag{16}$$
$$\dot{p}_z = 0\ ,\ \dot{p}_u = 0\ ,\ \dot{p}_v = 0. \tag{17}$$

The conservation laws of the dynamical system with Hamiltonian (14) are the momentum $p_z,\ p_u$ and p_v. The original Hamiltonian (3) is recovered when V

$$V\left(x\right)=\frac{1}{2}\alpha F_1\left(x\right)p_z^2+F_2\left(x\right)p_u^2+p_up_v \tag{18}$$

and

$$p_up_v-h_{1+3}=h. \tag{19}$$

Finally, the corresponding Eisenhart metric has the line element

$$ds_{(1+3)}^2=dx^2+\frac{1}{\alpha F_1\left(x\right)}dz^2+2dudv-2F_2\left(x\right)du^2. \tag{20}$$

2.2.2 Equivalent Hamiltonian $\hat{H}_{1+3}$

We introduce the extended Hamiltonian function as follows

$$\hat{H}_{1+3}=\frac{1}{2}p_x^2+\frac{1}{2}\alpha F_1\left(x\right)p_z^2+V_1\left(x\right)p_u^2+V_2\left(x\right)p_up_v=\hat{h}_{1+3} \tag{21}$$

with equations of motion

$$\dot{x}=p_x\ ,\ \dot{z}=\alpha F_1\left(x\right)p_z\ ,\ \dot{u}=2V_1\left(x\right)p_u+V_2\left(x\right)p_v\ ,\ \dot{v}=V_2\left(x\right)p_u\ , \tag{22}$$

$$\dot{p}_x=-\left(\frac{1}{2}\alpha F_1\left(x\right)_{,x}p_z^2+V_1\left(x\right)_{,x}p_u^2+V_2\left(x\right)_{,x}p_up_v\right)\ , \tag{23}$$

$$\dot{p}_z=0\ ,\ \dot{p}_u=0\ ,\ \dot{p}_v=0. \tag{24}$$

The corresponding Eisenhart metric is defined as

$$d\hat{s}_{(1+3)}^2=dx^2+\frac{1}{\alpha F_1\left(x\right)}dz^2+\frac{2}{V_2\left(x\right)}dudv-2\frac{V_1\left(x\right)}{\left(V_2\left(x\right)\right)^2}du^2. \tag{25}$$

and the original Hamiltonian system (3) is recovered for

$$V\left(x\right)=\frac{1}{2}\alpha F_1\left(x\right)p_z^2+V_1\left(x\right)p_u^2+V_2\left(x\right)p_up_v \tag{26}$$

and $\hat{h}_{1+3}=h$.

Without loss of generality in the latter extension we can assume that $F_1\left(x\right)=1$, such that the original system to be recovered when

$$V\left(x\right)=V_1\left(x\right)p_u^2+V_2\left(x\right)p_up_v \tag{27}$$

and

$$\frac{1}{2}\alpha p_z^2=h\ \text{ and }\hat{h}_{1+3}=0. \tag{28}$$

Hence, the latter dynamical systems to describe the null geodesics of the line element (25).

3 Geometric Linearization of Newtonian Systems

In the previous section, we established the geometric representation of (1). Therefore, the global linearization problem of (1) is equivalent to the linearization of the corresponding geodesic equations for each of the extended Hamiltonians. Before search into the analysis of this problem, we first discuss the application of conformal transformations for null geodesic equations. Specifically, we focus on autonomous Hamiltonian systems where the value of the Hamiltonian conservation law is constrained to be zero.

Definition 1 The two metric tensors $g_{ij}\left(x^k\right)$, $\bar{g}_{ij}\left(x^k\right)$ are conformally related if and only if there exist a set of coordinates where $\bar{g}_{ij} = N^2\left(x^k\right) g_{ij}$. If g_{ij} is the flat space, then $\bar{g}_{ij}$ is a conformally flat space.

Lemma 2 *Consider the Riemannian space with metric $g_{ij}\left(x^k\right)$ and of dimension $n = \dim g_{ij}$. For $n = 2$, the space g_{ij} is always conformally flat, and it is flat if the Ricci scalar is zero. When $n = 3$, the Riemannian space is conformally flat if and only if the Cotton-York tensor is zero. The Cotton-York tensor is expressed in terms of the Riemann tensor and of the Ricci scalar as follows*

$$C_{\mu\nu\kappa} = R_{\mu\nu;\kappa} - R_{\kappa\nu;\mu} + \frac{1}{4}\left(R_{;\nu} g_{\mu\kappa} - R_{;\kappa} g_{\mu\nu}\right), \tag{29}$$

Finally, for $n > 3$, the space is conformally flat if and only if the Weyl tensor is zero. The latter tensor is defined as

$$C_{\mu\nu\kappa\lambda} = R_{\mu\nu\kappa\lambda} - \frac{2}{n-2}\left(g_{\mu[\kappa} R_{\lambda]\nu} - g_{\nu[\kappa} R_{\lambda]\mu}\right) + \frac{2}{(n-1)(n-2)} R g_{\mu[\kappa} g_{\lambda]\nu}. \tag{30}$$

Consider now the geodesic Lagrangian for the Riemannian space with metric g_{ij}

$$L\left(x^k, \dot{x}^k\right) = \frac{1}{2} g_{ij}\left(x^k\right) \dot{x}^i \dot{x}^j, \tag{31}$$

where $\dot{x} = \frac{dx}{dt}$. The Action reads

$$S = \int dt\left(L\left(x^k, \dot{x}^k\right)\right) = \int dt\left(\frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j\right). \tag{32}$$

Under the change of variable $d\tau = N^2\left(x^i\right) dt$,we introduce the conformal Lagrangian

$$\bar{L}\left(x^k, x'^k\right) = \frac{1}{2} N^2\left(x^k\right) g_{ij} x'^i x'^j, \tag{33}$$

in which $x'^i = \frac{dx}{d\tau}$.

The two Lagrangians $L\left(x^k, \dot{x}^k\right)$ and $\bar{L}\left(x^k, x'^k\right)$ describe the geodesic equations for the two conformal related metrics $g_{ij}\left(x^k\right)$ and $\bar{g}_{ij}\left(x^k\right)$. For the geodesic equations of conformal Lagrangians we have the following Theorem.

Theorem 3 *The Euler-Lagrange equations for two conformal Lagrangians transform covariantly under the conformal transformation relating the Lagrangians if and only if the Hamiltonian vanishes.*

This implies that null geodesics remain invariant under conformal transformations. Therefore, the following corollary easily follows.

Corollary 4 *The null geodesic equations of conformally flat Riemannian spaces can be linearized. There exist a coordinate system where the geodesic equations can be written in the form of the free particle.*

In Appendix A we demonstrate the application of the latter corollary.

We consider the Eisenhart metrics for the corresponding extended Hamiltonian functions H_{1+1}, H_{1+2}, H_{1+3} and $\hat{H}_{1+3}$.

For the H_{1+1} we impose that the energy density is not zero; consequently, in order the geodesic equations to be linearized the metric (5) should be the flat space. Thus, the two-dimensional space (5) is the flat space if and only if the Ricci scalar is zero, i.e.

$$2V_{,xx}V - 3\left(V_{,x}\right)^2 = 0, \tag{34}$$

with solution $V_A(x) = \frac{V_0}{(x-x_0)^2}$. Without loss of generality we assume $x_0 = 0$.

Moreover, for the rest Hamiltonian functions H_{1+2}, H_{1+3} and $\hat{H}_{1+3}$, we assume that they describe null geodesics. Hence, we require the Eisenhart metrics (13), (20) and (25) to be conformally flat.

Thus, the three-dimensional line element (13) describes a conformally flat geometry if and only if

$$V_{,xxx} = 0 \tag{35}$$

from where it follows $V_B(x) = \frac{\omega}{2}(x - x_0)^2 + \omega_0$.

Furthermore, the four-dimensional line element (20) describes a conformally flat geometry when the free functions $F_1(x)$, $F_2(x)$ are constraint by the following system of differential equations,

$$2F_{1,xx}F - 3\left(F_{1,x}\right)^2 = 0, \tag{36}$$

$$2F_{2,xx}F_1 + F_{2,x}F_{1,x} = 0. \tag{37}$$

The solution of the latter system is $F_1(x) = \frac{1}{\left(F_1^0 x - F_1^1\right)^2}$, $F_2(x) = F_2^0 + F_2^1\left(x - \frac{F_1^1}{F_1^0}\right)^2$. Hence, by replacing in (18) and eliminate the non-essential constants we end with the Ermakov potential with oscillator $V_C(x) = \frac{\omega}{2}x^2 + \frac{V_0}{x^2}$.

Finally, for the line element (25) we assume without loss of generality that $F_1(x) = 1$, hence, the space is conformally flat when

$$3V_2\left(V_{1,xx}V_2 - 3V_{1,x}V_{2,x}\right) + 6V_1\left(V_{2,x}\right)^2 = 0, \tag{38}$$

$$V_{2,xx}V_2 - \left(V_{2,x}\right)^2 = 0. \tag{39}$$

Hence, $V_1(x) = V_1^0 e^{\lambda x} + V_1^1 e^{2\lambda x}$ and $V_2(x) = V_2^0 e^{\lambda x}$. Consequently, the Morse potential $V(x) = V_1^0 e^{\lambda x} + V_2^0 e^{2\lambda x}$ follows.

The results are summarized in the following theorem.

Theorem 5 *The Newtonian system (1), with $F(x) = -V(x)_{,x}$ can be written as the free particle $\frac{d^2X}{dT^2} = 0$, for the potentials: (A) Ermakov potential $V_A(x) = \frac{V_0}{x^2}$; (B) the oscillator $V_B(x) = \frac{\omega}{2}x^2$; (C) the Ermakov potential with oscillator $V_C(x) = \frac{\omega}{2}x^2 + \frac{V_0}{x^2}$ and (D) the Morse potential $V(x) = V_1^0 e^{\lambda x} + V_2^0 e^{2\lambda x}$, via the Eisenhart lift and the equivalent Hamiltonians H_{1+1}, H_{1+2}, H_{1+3} and $\hat{H}_{1+3}$ respectively.*

The proof of the latter theorem is presented in Appendix A. Let us now demonstrate the latter result with some applications.

Consider the Hamiltonian H_{1+1} for the Ermakov potential $V_A(x) = \frac{V_0}{x^2}$. That is,

$$H_{1+1} \equiv \frac{1}{2}p_x^2 + \frac{\alpha V_0}{2x^2}p_z^2, \tag{40}$$

We introduce the new variables [52]

$$x=\sqrt{X^2+Y^2}\ ,\ z=\frac{1}{\sqrt{\alpha V_0}}\arctan\left(\frac{Y}{X}\right), \tag{41}$$

then, Hamiltonian (40) becomes

$$H_{1+1}=\frac{1}{2}\left(p_X^2+p_Y^2\right), \tag{42}$$

which is the Hamiltonian for the free particle. In a similar way we can linearize and the rest Eisenhart metrics.

The solution of Hamilton's (42) reveal four integration constants, while the original system process only two integration constants. Due to the constraints $h_{1+1}=h$ and $\alpha\left(p_z^0\right)^2=2$, we end with one free integration constant. As we discuss before that third integration constant is related with parameter z and plays no role in the solution for the original dynamical system.

Assume now the Hamiltonian function H_{1+2} with the quadratic potential $V_B(x)=\frac{\omega}{2}x^2$, i.e.

$$H_{1+2}\equiv\frac{1}{2}p_x^2+\frac{\omega}{2}x^2p_u^2+p_up_v. \tag{43}$$

Under the action of the point transformation

$$x=\frac{X}{\sqrt{1+\bar{Z}^2}}\ ,\ u=\frac{1}{\sqrt{\omega}}\arctan\bar{Z}\ ,\ v=\sqrt{\omega}\left(2Y-\bar{Z}+\frac{\bar{Z}}{\left(1+\bar{Z}^2\right)}R^2\right)\ ,\ \bar{Z}=Y+Z \tag{44}$$

we end up with the Hamiltonian

$$H_{1+2}=\left(1+\bar{Z}^2\right)\left(p_X^2+p_Y^2-p_Z^2\right). \tag{45}$$

Transformation $(x,u)\rightarrow\left\{X,\bar{Z}\right\}$ is nothing else than the usual transformation provided by the Lie symmetry analysis which relates the oscillator to the free particle [51].

We continue our analysis by introducing the Hamiltonian function

$$\hat{H}_{1+3}=\frac{1}{2}p_x^2+\frac{1}{2}p_z^2+V_{10}e^{\lambda x}p_u^2+V_{20}e^{\lambda x}p_up_v \tag{46}$$

which leads to the Newtonian system (1) for the exponential potential $V(x)=V_1^0e^{\lambda x}$. Under the change of variables

$$x=\frac{2}{\lambda}\ln\left(\frac{\lambda\sqrt{V_{10}Z^2+V_{20}\left(WX-YZ\right)}}{\sqrt{2}V_{20}W}\right)\ ,\ z=-\frac{2i}{\lambda}\ln\left(\frac{\lambda\sqrt{V_{10}Z^2+V_{20}\left(WX-YZ\right)}}{\sqrt{2}V_{20}W}\right), \tag{47}$$

$$u=\frac{Z}{W}\ ,\ v=\frac{Y}{W}, \tag{48}$$

the latter Hamiltonian function reads

$$\hat{H}_{1+3}=e^{-2i\lambda z}\left(-V_{20}p_Xp_W+2V_{10}p_Y^2+2V_{20}p_YP_Z\right)=0, \tag{49}$$

where the resulting equations of motion are linear, that is,

$$\ddot{X}=0\ ,\ \ddot{Y}=0\ ,\ \ddot{Z}=0\ ,\ \ddot{W}=0. \tag{50}$$

4 Conclusions

We conducted a study on the linearization of the equation of motion for a particle moving along a line under the influence of an autonomous force, within the framework of a Hamiltonian dynamical system. To achieve this, we employed the strategy of the Eisenhart lift to formulate a system of geodesic equations that are equivalent to the original problem. By defining various sets of equivalent geodesic Hamiltonian systems and working within the framework of Riemannian geometry, we were able to determine the functional forms for the force term. This allowed us to identify cases where the geodesic equations could be linearized, effectively transforming them into the equations of motion for a free particle in a plane geometry.

Our investigation revealed that for the Ermakov potential, the oscillator, and the Morse potential, we could define a higher-dimensional geometry where the equivalent geodesic equations could be linearized. Consequently, for these potentials, the Newtonian system under our consideration becomes equivalent to that of a free particle, under a change of variables.

This method opens up new directions for studying classical systems and for the linearization of differential equations. In future work, we aim to extend this analysis to higher-dimensional dynamical systems.

A One-Dimensional Constraint System

Consider the null geodesics for the conformally flat line element (5). We shall see that for arbitrary potential the dynamical system can be global linearized.

We introduce the change of variables $dx = F(X)\,dX$, thus, the line element is

$$ds^2_{(1+1)} = F^2(X)\,dX^2 + \frac{1}{\alpha V\left(\int F(X)\,dX\right)}dz^2, \tag{51}$$

we select function $F(X)$ such that $F^2(X) = \frac{1}{V\left(\int F(X)dX\right)}$. Consequently, Hamiltonian (4) for the null geodesics reads

$$H_{1+1} \equiv V\left(\int F(X)\,dX\right)\left(p_X^2 + \frac{1}{\alpha}p_z^2\right) = 0. \tag{52}$$

with equations of motion

$$\dot{X} = V\left(\int F(X)\,dX\right)p_X\ ,\ \dot{z} = V\left(\int F(X)\,dX\right)p_z\ , \tag{53}$$

$$\dot{p}_X = V\left(\int F(X)\,dX\right)_{,X}\left(p_X^2 + \frac{1}{\alpha}p_z^2\right)\ ,\ \dot{p}_Z = 0. \tag{54}$$

With the use of the constraint (52) it follows

$$\dot{X} = V\left(\int F(X)\,dX\right)p_X\ ,\ \dot{z} = V\left(\int F(X)\,dX\right)p_z\ , \tag{55}$$

$$\dot{p}_X = 0\ ,\ \dot{p}_Z = 0. \tag{56}$$

We consider the new independent variable $d\tau = V\left(\int F(X)\,dX\right)dt$, from where it follows

$$X' = p_X\ ,\ z' = p_z\ ,\ p_X' = 0\ ,\ p_z' = 0, \tag{57}$$

or in the equivalently form

$$X'' = 0\ ,\ z'' = 0. \tag{58}$$

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Author Contributions This is a single author work.

Data Availability No datasets were generated or analysed during the current study.

Declarations

Competing Interests The authors declare no competing interests.

References

1. Lie, S.: Theorie der Transformationsgrupprn:, vol. I. Chelsea, New York (1970)
2. Lie, S.: Theorie der Transformationsgrupprn:, vol. II. Chelsea, New York (1970)
3. Lie, S.: Theorie der Transformationsgrupprn:, vol. III. Chelsea, New York (1970)
4. N.H. Ibragimov: CRC Handbook of Lie Group Analysis of Differential Equations, Volume I: Symmetries, Exact Solutions, and Conservation Laws. CRS Press LLC, Florida (2000)
5. Bluman, G.W., Kumei, S.: Symmetries of Differential Equations. Springer-Verlag, New York (1989)
6. Stephani, H.: Differential Equations: Their Solutions Using Symmetry. Cambridge University Press, New York (1989)
7. Olver, P.J.: Applications of Lie Groups to Differential Equations. Springer-Verlag, New York (1993)
8. Chou, K.S., Qu, C.Z.: Optimal systems and group classification of (1+2)-dimensional heat equation. Acta Appl. Math. **83**, 257 (2004)
9. Mahomed, F.M.: Symmetry group classification of ordinary differential equations. Math. Methods Appl. Sci. **30**, 1995 (2007)
10. Meleshko, S.V., Shapeev, V.P.: Nonisentropic solutions of simple wave type of the gas dynamics equations. J. Nonl. Math. Phys. **18**, 195 (2011)
11. Mahomed, F.M., Leach, P.G.L.: The lie algebra SL (3, R) and linearization. Quaest. Math. **12**, 121 (1989)
12. Moyo, S., Leach, P.G.L.: Symmetry properties of autonomous integrating factors. J. Math. Anal. Appl. **252**, 840 (2000)
13. Leach, P.G.L.: Sl(3, R) and the repulsive oscillator. J. Phys. A: Math. Gen. **13**, 1991 (1980)
14. Paliathanasis, A., Tsamparlis, M., Mustafa, M.T.: Classification of the Lie and Noether point symmetries for the wave and the Klein-Gordon equations in pp-wave spacetimes. Int. J. Geom. Meth. Mod. Phys. **12**, 1550033 (2015)
15. Kara, A.H., Mahomed, F.M.: Classification of first-order Lagrangians on the line. Int. J. Theor. Phys. **34**, 2267 (1995)
16. Chesnokov, A.A.: Symmetries and exact solutions of the shallow water equations for a two-dimensional shear flow. J. Appl. Mech. Techn. Phys. **49**, 737 (2008)
17. Chesnokov, A.A.: Symmetries and exact solutions of the rotating shallow-water equations. Eur. J. Appl. Math. **20**, 461 (2009)
18. Paliathanasis, A.: One-dimensional optimal system for 2d rotating ideal gas. Symmetry **11**, 1115 (2019)
19. Xin, X., Zhang, L., Xia, Y., Liu, H.: Nonlocal symmetries and exact solutions of the (2+1)-dimensional generalized variable coefficient shallow water wave equation. Appl. Math. Lett. **94**, 112 (2019)
20. Dorodintsyn, V.A., Kaptsov, E.I.: Invariant conservative difference schemes for shallow water equations in Eulerian and Lagrangian coordinates. J. Math. Phys. **62**, 083508 (2021)
21. Jamal, S., Kara, A.H.: New higher-order conservation laws of some classes of wave and Gordon-type equations. Nonlinear Dyn. **57**, 97 (2012)
22. Camci, U., Jamal, S., Kara, A.H.: Invariances and conservation laws based on some FRW universes. Int. J. Theor. Phys. **53**, 1483 (2014)

23. Gwaxa, B., Jamal, S., Johnpillai, A.G.: On the conservation laws, Lie symmetry analysis and power series solutions of a class of third-order polynomial evolution equations, Arabian. J. Math. **12**, 553 (2023)
24. Bokhari, A.H., Mahomed, F.M., Zaman, F.D.: Symmetries and integrability of a fourth-order Euler-Bernoulli beam equation. J. Math. Phys. **51**, 053517 (2010)
25. Ahmada, A., Bokhari, A.H., Kara, A.H., Zaman, F.D.: Symmetry classifications and reductions of some classes of (2+1)-nonlinear heat equation. J. Math. Anal. Appl. **339**, 175 (2008)
26. Andriopoulos, K., Leach, P.G.L., Flessas, G.P.: Complete symmetry groups of ordinary differential equations and their integrals: some basic considerations. J. Math. Anal. Appl. **262**, 256 (2001)
27. Waho Soh, C., Mahomed, F.M.: Linearization criteria for a system of second order ordinary differential equations. Int. J. Non Linear Mech. **36**, 671 (2001)
28. Qadir, A.: Linearization of ordinary differential equations by using geometry. Sigma **3**, 1 (2007)
29. Dutt, H.M., Qadir, A.: Classification of scalar third order ordinary differential equations linearizable via generalized contact transformations
30. Ali, S., Mahomed, F.M., Qadir, A.: Linearizability criteria for systems of two second-order differential equations by complex methods. Nonlinear Dyn. **66**, 77 (2011)
31. Qadir, A.: Linearization: geometric, complex, and conditional. J. Appl. Math. **2012**, 303960 (2012)
32. Aminova, A. V., Aminov, N. A.-M.: Projective geometry of systems of differential equations: general conceptions, Tensor N.S. 62. 65 (2000)
33. Aminova, A.V., Aminov, N.A.-M.: Projective geometry of systems of second-order differential equations. Sbornik Math. **197**, 951 (2006)
34. Feroze, T., Mahomed, F.M., Qadir, A.: The connection between isometries and symmetries of geodesic equations of the underlying spaces. Nonlinear Dyn. **45**, 65 (2006)
35. Tsamparlis, M., Paliathanasis, A.: Lie symmetries of geodesic equations and projective collineations. Nonlinear Dyn. **62**, 203 (2010)
36. Morozov, V.V.: Classification of six-dimensional nilpotent Lie algebras. Izvestia Vysshikh Uchebn Zavendeniĭ Matematika **5**, 161 (1958)
37. Mubarakzyanov, G.M.: On solvable Lie algebras, Izvestia Vysshikh Uchebn Zavendeniĭ. Matematika **32**, 114 (1963)
38. Mubarakzyanov, G.M.: Classification of real structures of five-dimensional Lie algebras. Izvestia Vysshikh Uchebn Zavendeniĭ Matematika **34**, 99 (1963)
39. Mubarakzyanov, G.M.: Classification of solvable six-dimensional Lie algebras with one nilpotent base element. Izvestia Vysshikh Uchebn Zavendeniĭ Matematika **35**, 104 (1963)
40. Eisenhart, L.P.: Dynamical trajectories and geodesics. Annals. Math. **30**, 591 (1928)
41. Cariglia, M., Galajinsky, A., Gibbons, G.W., Horvathy, P.A.: Cosmological aspects of the Eisenhart-Duval lift. EPJC **78**, 314 (2018)
42. Kan, N., Aoyama, T., Hasegawa, T., Shiraishi, K.: Eisenhart-Duval lift for minisuperspace quantum cosmology. Phys. Rev. D **104**, 086001 (2021)
43. Sen, A., Parida, B.K., Dhasmana, S., Silagadze, Z.K.: Eisenhart lift of Koopman-von Neumann mechanics. J. Geom. Phys. **185**, 104732 (2023)
44. Cariglia, M., Alves, F.K.: The Eisenhart lift: a didactical introduction of modern geometrical concepts from Hamiltonian dynamics. Eur. J. Phys. **36**, 025018 (2015)
45. Cariglia, M., Duval, C., Gibbons, G.W., Horvathy, P.A.: Eisenhart lifts and symmetries of time-dependent systems. Ann. Phys. **373**, 631 (2016)
46. Paliathanasis, A.: Classical and Quantum solutions in Scalar field cosmology via the Eisenhart lift and linearization (2024)
47. Dhasmana, S., Sen, A., Silagadze, Z.K.: Equivalence of a harmonic oscillator to a free particle and Eisenhart lift. Ann. Phys. **434**, 168623 (2021)
48. Duval, C., Burdet, G., Kunzle, H.P., Perrin, M.: Bargmann structures and Newton-Cartan theory. Phys. Rev. D **1841**, 31 (1985)
49. Duval, C., Horváthy, P.A., Palla, L.: Conformal properties of chern-simons vortices in external fields. Phys. Rev. D **50**, 6658 (1994)
50. Zhang, P.M., Elbistan, M., Horváthy, P.A.: Particle motion in circularly polarized vacuum pp waves. Class. Quantum Grav. **39**, 035008 (2022)
51. Sarlet, W., Mahomed, F.M., Leach, P.G.L.: Symmetries of non-linear differential equations and linearisation. J. Phys A. Math. Gen. **20**, 277 (1987)
52. Paliathanasis, A.: Solving Nonlinear Second-Order ODEs via the Eisenhart Lift and Linearization. Axioms **13**, 331 (2024)

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